8.5.1

Bestem konvergensradien til

a) $\sum_{n=0}^{\infty} \frac{n}{3^n} x^n$

Forholdstesten:

$$\lim_{n \to \infty} \left| \frac{\frac{n+1}{3^{n+1}} x^{n+1}}{\frac{n}{3^n} x^n} \right| = \lim_{n \to \infty} \left| \frac{(n+1) \cdot \cancel{3^n} \cdot x^{\cancel{n+1}}}{n \cdot 3^{\cancel{n+1}} \cdot \cancel{x^n}} \right|$$

$$= \lim_{n \to \infty} \left| \frac{n+1}{n} \cdot \frac{x}{3} \right| = \lim_{n \to \infty} \frac{(n+1):n}{(n):n} \frac{|x|}{3}$$

$$= \lim_{n \to \infty} \frac{1+\frac{1}{n}}{1} \frac{|x|}{3} = \frac{|x|}{3} < 1 \qquad |x| < \boxed{3}$$

$(-3, 3)$

Konvergensradien er 3.

b) $\sum \frac{9^n}{n+1} x^{2n}$

Forholdstesten

$$\lim_{n\to\infty} \left| \frac{\frac{9^{n+1}}{n+2} \cdot x^{2(n+1)}}{\frac{9^n}{n+1} x^{2n}} \right| = \lim_{n\to\infty} \left| \frac{9^{n+1} (n+1) x^{2n+2}}{\cancel{9^n} (n+2) \cancel{x^{2n}}} \right|$$

$$= \lim_{n\to\infty} \left| 9 \cdot \frac{n+1}{n+2} \cdot x^2 \right| = \lim_{n\to\infty} 9 \cdot \frac{(n+1):n}{(n+2):n} \cdot |x|^2$$

$$= \lim_{n\to\infty} \frac{1+\frac{1}{n}}{1+\frac{2}{n}} \cdot 9|x|^2 = 9|x|^2 < 1 \qquad |x|^2 < \frac{1}{9}$$

$$|x| < \boxed{\frac{1}{3}}$$

Konvergensradien er $\frac{1}{3}$.

c) $\sum_{n=1}^{\infty} \frac{1}{n^n} x^n$

Forholdstesten: $\lim_{n\to\infty} \left| \frac{\frac{1}{(n+1)^{n+1}} x^{n+1}}{\frac{1}{n^n} \cancel{x^n}} \right| = \lim_{n\to\infty} \boxed{\frac{n^n}{(n+1)^{n+1}}} |x|$

Rot-testen:

$$\lim_{n\to\infty} \sqrt[n]{|a_n|} = \lim_{n\to\infty} \sqrt[n]{\left|\frac{1}{n^n}x^n\right|} = \lim_{n\to\infty} \sqrt[n]{\frac{|x|^n}{n^n}}$$

$$= \lim_{n\to\infty} \frac{\sqrt[n]{|x|^n}}{\sqrt[n]{n^n}} = \lim_{n\to\infty} \boxed{\frac{|x|}{n}} = 0 < 1 \quad \text{Alltid sant.}$$

Konvergensradien er ∞.

8.5.4 Bestem konvergensområdet til

a) $\displaystyle\sum_{n=0}^{\infty} \frac{(3x-2)^n}{n+1}$

Forholdstesten:

$$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left|\frac{\frac{(3x-2)^{n+1}}{n+2}}{\frac{(3x-2)^n}{n+1}}\right| = \lim_{n\to\infty} \left|\frac{(3x-2)^{\cancel{n+1}}(n+1)}{\cancel{(3x-2)^n}(n+2)}\right|$$

$$= \lim_{n\to\infty} |3x-2| \frac{(n+1):n}{(n+2):n} = \lim_{n\to\infty} |3x-2| \frac{1+\boxed{\frac{1}{n}}}{1+\boxed{\frac{2}{n}}} = |3x-2| < 1$$

$$-1 < 3x-2 < 1 \qquad 1 < 3x < 3 \qquad \frac{1}{3} < x < 1$$

Vet at rekka konvergerer når $x \in \langle \frac{1}{3}, 1 \rangle$

Hva med endepunktene? Hva med $x = \frac{1}{3}$ og $x = 1$

$$\sum_{n=0}^{\infty} \frac{(3x-2)^n}{n+1}$$

$x=\frac{1}{3}$: $\sum_{n=0}^{\infty} \frac{(3\cdot\frac{1}{3}-2)^n}{n+1} = \sum_{n=0}^{\infty} \frac{(-1)^n}{n+1}$ $\quad \frac{1}{n+1} \to 0$

Konvergerer i følge alternerende rekketest.

$x=1$: $\sum_{n=0}^{\infty} \frac{(3\cdot 1-2)^n}{n+1} = \sum_{n=0}^{\infty} \frac{1}{n+1} = 1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\frac{1}{5}$

$= \sum_{n=1}^{\infty} \frac{1}{n}$ Divergerer i følge p-testen. (Harmonisk rekke)

Konvergensområdet blir $\left[\frac{1}{3}, 1\right\rangle$

b) $\sum_{n=0}^{\infty} \frac{x^n}{(n+1)4^n}$ Forholdstest:

$$\lim_{n\to\infty} \left| \frac{\frac{x^{n+1}}{(n+2)4^{n+1}}}{\frac{x^n}{(n+1)4^n}} \right| = \lim_{n\to\infty} \left| \frac{(n+1)\cancel{4^n}\, x^{n+1}}{(n+2)\, 4^{n+1}\, \cancel{x^n}} \right|$$

$$= \lim_{n\to\infty} \frac{|x|}{4} \cdot \frac{(n+1):n}{(n+2):n} = \lim_{n\to\infty} \frac{|x|}{4} \frac{1+\frac{1}{n}}{1+\frac{2}{n}}$$

$$= \frac{|x|}{4} < 1 \qquad |x| < 4$$

$$-4 < x < 4$$

Må sjekke $x=4$ og $x=-4$.

$x=4$

$$\sum_{n=0}^{\infty} \frac{\cancel{4^n}}{(n+1)\cancel{4^n}} = \sum_{n=0}^{\infty} \frac{1}{n+1} = \sum_{n=1}^{\infty} \frac{1}{n}$$ Divergerer.

$(-4)^n = (-1)^n \cdot 4^n$

$x=-4$

$$\sum_{n=0}^{\infty} \frac{(-4)^n}{(n+1)4^n} = \sum_{n=0}^{\infty} \frac{(-1)^n \cancel{4^n}}{(n+1)\cancel{4^n}} = \sum_{n=0}^{\infty} \frac{(-1)^n}{n+1}$$ Konvergerer i ~~følge~~ alternerende rekketest.

Konvergensområdet er $[-4, 4\rangle$

8.5.6 Deriver potensrekkene:

a) $$\left(\sum_{n=0}^{\infty} \frac{2^n-1}{(n-1)!} x^n\right)' = \sum_{n=0}^{\infty} \frac{2^n-1}{(n-1)!} (x^n)'$$

$$= \sum_{n=0}^{\infty} \frac{2^n-1}{(n-1)!} \boxed{n} \cdot x^{n-1}$$

① $(n=0)$-leddet forsvinner. $\sum_{n=1}^{\infty} \frac{2^n-1}{(n-1)!} \cdot n \cdot x^{n-1}$

② vil ha x^n, ikke x^{n-1}. $m = n-1 \Rightarrow n = m+1$

$$\sum_{m=0}^{\infty} \frac{2^{m+1}-1}{m!} \cdot (m+1) x^m = \sum_{n=0}^{\infty} \frac{2^{n+1}-1}{n!} \cdot (n+1) x^n$$

b) $\left(\sum_{n=0}^{\infty} \frac{3^n x^{2n}}{n!}\right)' = \sum_{n=0}^{\infty} \frac{3^n}{n!} 2n \cdot x^{2n-1} = \underline{\underline{\sum_{n=1}^{\infty} \frac{3^n}{n!} 2n\, x^{2n-1}}}$

8.5.7. Finn rekkeutviklingen til hvert integral.

a) $\int_0^x \sum_{n=0}^{\infty} \frac{t^{2n}}{n!} dt = \sum_{n=0}^{\infty} \int_0^x \frac{t^{2n}}{n!} dt$ $\qquad \int t^n dt = \frac{1}{n+1} t^{n+1}$

$= \sum_{n=0}^{\infty} \frac{1}{n!} \int_0^x t^{2n} dt = \sum_{n=0}^{\infty} \frac{1}{n!} \cdot \left[\frac{1}{2n+1} t^{2n+1}\right]_0^x$

$= \sum_{n=0}^{\infty} \frac{1}{n!} \cdot \frac{1}{2n+1} \cdot x^{2n+1} = \sum_{n=0}^{\infty} \frac{x^{2n+1}}{(2n+1)\, n!}$

b) $\int_0^{2x} \frac{e^t - 1}{t} dt$

$e^t = \sum_{n=0}^{\infty} \frac{t^n}{n!} = 1 + t + \frac{t^2}{2} + \frac{t^3}{6} + \dots$

$e^t - 1 = t + \frac{t^2}{2} + \frac{t^3}{6} + \dots = \sum_{n=1}^{\infty} \frac{t^n}{n!}$

$\frac{e^t - 1}{t} = \frac{\sum_{n=1}^{\infty} \frac{t^n}{n!}}{t} = \sum_{n=1}^{\infty} \frac{t^n}{n!\, \cancel{t}} = \sum_{n=1}^{\infty} \frac{t^{n-1}}{n!}$

$$\int_0^{2x} \frac{e^t - 1}{t}\,dt = \int_0^{2x} \sum_{n=1}^{\infty} \frac{t^{n-1}}{n!}\,dt = \sum_{n=1}^{\infty} \frac{1}{n!}\int_0^{2x} t^{n-1}\,dt$$

$$= \sum_{n=1}^{\infty} \frac{1}{n!}\cdot\frac{1}{n}\Big[t^n\Big]_0^{2x} = \sum_{n=1}^{\infty} \frac{(2x)^n}{n!\cdot n}$$

$$= \sum_{n=1}^{\infty} \frac{2^n x^n}{n\cdot n!}$$

Hva er $\int_0^2 \frac{e^t - 1}{t}\,dt$? $= \sum_{n=1}^{\infty} \frac{2^n}{n\cdot n!} = 2 + \frac{4}{4} + \frac{8}{18} + \ldots$

$$\approx 3.4$$

8.5.8 Hvilke funksjoner har disse potensrekkene? Thm 8.5.10

a) $\sum_{n=0}^{\infty} \frac{x^{2n}}{n!} = \sum_{n=0}^{\infty} \frac{\overbrace{(x^2)}^{y}{}^n}{n!}$ $\qquad e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$

$$= \sum_{n=0}^{\infty} \frac{y^n}{n!} = e^y = \underline{\underline{e^{x^2}}}$$

b) $\sum_{n=0}^{\infty} 2^n (n+1) x^n = \sum_{n=0}^{\infty} (n+1)(2x)^n$ $\qquad \frac{1}{1-x} = \sum_{n=0}^{\infty} x^n$

$$\left(\sum_{n=0}^{\infty} x^n\right)' = \sum_{n=0}^{\infty} n\cdot x^{n-1} = \sum_{n=1}^{\infty} n\cdot x^{\overbrace{\boxed{n-1}}^{m}} = \sum_{m=0}^{\infty} (m+1)x^m$$

Må finne $\left(\frac{1}{1-x}\right)'$

$$\left(\frac{1}{1-x}\right)' = \left((1-x)^{-1}\right)' = -1\cdot(1-x)^{-2}\cdot(-1)$$

$$= \frac{1}{(1-x)^2}$$

$$\sum_{n=0}^{\infty} 2^n(n+1)x^n = \sum_{n=0}^{\infty}(n+1)(2x)^n = \frac{1}{(1-2x)^2}$$

c)

$$\sum_{n=0}^{\infty}(-3)^{n+1}\frac{x^{2n}}{(2n)!} \qquad \cos x = \sum_{n=0}^{\infty}\frac{(-1)^n}{(2n)!}x^{2n}$$

Må prøve å gjøre $(-3)^{n+1}x^{2n} = k\cdot(-1)^n y^{2n}$.

$$(-3)^{n+1}x^{2n} = -3\cdot(-3)^n x^{2n} = -3\cdot(-1)^n 3^n x^{2n}$$

$$= -3\cdot(-1)^n\left((\sqrt{3})^2\right)^n x^{2n} = -3\cdot(-1)^n(\sqrt{3})^{2n}x^{2n}$$

$$= -3\cdot(-1)^n\cdot(\sqrt{3}x)^{2n}$$

Så

$$\sum_{n=0}^{\infty}(-3)^{n+1}\frac{x^{2n}}{(2n)!} = -3\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n)!}(\sqrt{3}x)^{2n} = \underline{\underline{-3\cdot\cos(\sqrt{3}x)}}.$$

8.5.111

a) Med utgangspunkt i potensrekken for sinus, forklar at

$$\int_0^x \boxed{\frac{\sin t}{t}}\, dt = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)\cdot(2n+1)!} x^{2n+1}$$

$$\sin x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!} = x - \frac{x^3}{6} + \frac{x^5}{120} - \dots$$

$$\frac{\sin t}{t} = \sum_{n=0}^{\infty} \frac{(-1)^n t^{\cancel{2n+1}}}{(2n+1)!\,\cancel{t}} = \sum_{n=0}^{\infty} \frac{(-1)^n t^{2n}}{(2n+1)!}$$

$$\int_0^x \frac{\sin t}{t} dt = \int_0^x \sum_{n=0}^{\infty} \frac{(-1)^n t^{2n}}{(2n+1)!} dt = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} \int_0^x t^{2n} dt$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} \cdot \frac{1}{2n+1} \left[t^{2n+1} \right]_0^x$$

$$= \sum \frac{(-1)^n x^{2n+1}}{(2n+1)(2n+1)!}$$

b) Bruk potensrekka til å finne en tilnærming til

$$\int_0^1 \frac{\sin t}{t} dt$$ med feil mindre enn 10^{-4}

$$\int_0^1 \frac{\sin t}{t}\,dt = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)(2n+1)!} \Big|_{}^{1} 1^{2n+1}$$

Feil mindre enn $\frac{1}{10\,000}$

$$= 1 - \frac{1}{18} + \frac{1}{600} - \frac{1}{35280} + \frac{1}{3265920}$$

$$= 0.946\boxed{0}8$$