8.3.3

a) Vis at

$$\frac{1}{(n+1)!} \leq \frac{1}{2^n} \quad \text{for alle } n \geq 1.$$

$$\Updownarrow$$

$$2^n \leq (n+1)!$$

Eksempel: $n = 5$

$$\begin{array}{l} 2^5 = \boxed{2} \cdot \boxed{2} \cdot \boxed{2} \cdot \boxed{2} \cdot \boxed{2} \\ \quad\quad\;\; \wedge \quad \wedge \quad \wedge \quad \wedge \quad \shortparallel \\ (5+1)! = \boxed{6} \cdot \boxed{5} \cdot \boxed{4} \cdot \boxed{3} \cdot \boxed{2} \end{array}$$

Derfor er $2^5 \leq (5+1)!$

$$2^n = \overbrace{2 \cdot 2 \cdot 2 \cdot \ \cdots \ \cdot 2}^{n \text{ tall}}$$

Alle er 2

$$(n+1)! = \underbrace{(n+1) \cdot n \cdot (n-1) \cdot \ \cdots \ \cdot 3 \cdot 2}_{n \text{ tall}}$$

Alle er større enn eller lik 2.

Derfor må $2^n \leq (n+1)!$

Må ha $\leq$ fordi $n=1$ gir $2^1 = (1+1)!$

Neste oppgave skal bruke at

$$\frac{1}{(n+1)!} \leq \frac{1}{2^n}$$

b) Bruk dette til å finne ut om

$$\sum_{n=1}^{\infty} \frac{1}{(n+1)!}$$

konvergerer eller divergerer.

Vet at $\frac{1}{(n+1)!} \leq \frac{1}{2^n}$.

Derfor må

$$\sum_{n=1}^{\infty} \frac{1}{(n+1)!} \leq \sum_{n=1}^{\infty} \frac{1}{2^n} = \sum_{n=1}^{\infty} \left(\frac{1}{2}\right)^n$$

Dette er en geometrisk rekke, så den konvergerer.
Derfor må også $\sum_{n=1}^{\infty} \frac{1}{(n+1)!}$ konvergere.

c) Vi vet at Eulertallet $e = \sum_{n=0}^{\infty} \frac{1}{n!}$. Vis at $e \leq 3$

Ser at $\sum_{n=0}^{\infty} \frac{1}{n!} = 1 + 1 + \frac{1}{2!} + \frac{1}{3!} + \dots$

og at $\sum_{n=1}^{\infty} \frac{1}{(n+1)!} = \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!} + \dots$

Betyr at $\sum_{n=0}^{\infty} \frac{1}{n!} = 1 + 1 + \boxed{\sum_{n=1}^{\infty} \frac{1}{(n+1)!}} \leq 2 + \sum_{n=1}^{\infty} \left(\frac{1}{2}\right)^n$

Må regne ut denne.

Vet at $\sum_{n=0}^{\infty} (k)^n = \frac{1}{1-k}$ når $-1 < k < 1$

Så $\sum_{n=0}^{\infty} \left(\frac{1}{2}\right)^n = \frac{1}{1-\frac{1}{2}} = 2$

$$= 1 + \boxed{\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \dots}$$

$\Uparrow$

Må bli 1.

Da har vi:

$$e = \sum_{n=0}^{\infty} \frac{1}{n!} = 1 + 1 + \sum_{n=1}^{\infty} \frac{1}{(n+1)!} \leq 2 + \sum_{n=1}^{\infty} \frac{1}{2^n}$$

$$= 2 + 1 = 3$$

8.4.1 Avgjør om rekkene konvergerer eller divergerer.

a) $\sum_{n=1}^{\infty} \boxed{\frac{1}{2\sqrt{n}}}$

Divergenstesten:

$$\lim_{n \to \infty} \frac{1}{2\sqrt{n}} = 0$$

Fremdeles ukjent om dette divergerer eller konvergerer.

Alternerende rekketest: Denne rekka er <u>ikke</u> alternerende, så fremdeles ingen konklusjon.

Forholdstesten:

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{\frac{1}{2\sqrt{n+1}}}{1/2\sqrt{n}} = \lim_{n\to\infty} \frac{1}{\cancel{2}\sqrt{n+1}} \cdot \frac{\cancel{2}\sqrt{n}}{1}$$

$$= \lim_{n\to\infty} \frac{\sqrt{n}}{\sqrt{n+1}} = \lim_{n\to\infty} \sqrt{\frac{n}{n+1}}$$

$$= \lim_{n\to\infty} \sqrt{\frac{1}{1+\boxed{1/n}}} = \sqrt{\frac{1}{1}} = 1.$$

Forholdstesten sier: Om svaret er mindre enn 1, konvergens

— " — større enn 1, divergens,

— " — lik 1, ukjent.

Huskeregel: Hvis a_n ser ut som „polynomer" delt på hverandre, vil forholdstestan gi 1 som svar.

Dvs n^k summer. I vårt tilfelle har vi $\frac{1}{2\sqrt{n}} = \frac{1}{2n^{1/2}}$

Vi vil typisk bruke p-testen på disse.

$$\sum \frac{1}{2\sqrt{n}} = \sum \frac{1}{2n^{1/2}} = \frac{1}{2} \cdot \boxed{\sum \frac{1}{n^{1/2}}} = \frac{1}{2} \cdot \sum \frac{1}{n^p}, \quad p = \frac{1}{2}$$

p-testen sier at dette konvergerer om $p > 1$, og divergerer om $p \leq 1$. Siden $\frac{1}{2} \leq 1$, vil dette divergere.

b) $\sum_{n=0}^{\infty} \frac{2^n}{(n+1)3^n}$

Divergenstesten:

$$\lim_{n\to\infty} \frac{2^n}{(n+1)3^n} = \lim_{n\to\infty} \frac{1}{n+1} \cdot \frac{2^n}{3^n} = \lim_{n\to\infty} \frac{1}{n+1} \cdot \left(\frac{2}{3}\right)^n$$

$$= \lim_{n\to\infty} \frac{\left(\frac{2}{3}\right)^n}{n+1} = 0$$

Divergenstesten gir ingen svar.

Alternerende rekketest: Rekka er ikke alternerende.

Forholdstesten:

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{\frac{2^{n+1}}{(n+2)3^{n+1}}}{\frac{2^n}{(n+1)3^n}} = \lim_{n\to\infty} \frac{2^{n+1}}{(n+2)3^{n+1}} \cdot \frac{(n+1)3^n}{2^n}$$

$$= \lim_{n\to\infty} \frac{2^{n+1}}{\cancel{2^n}} \cdot \frac{\cancel{3^n}}{3^{n+1}} \cdot \frac{n+1}{n+2}$$

$$= \lim_{n\to\infty} \frac{2}{3} \cdot \frac{n+1}{n+2} = \lim_{n\to\infty} \frac{2}{3} \frac{1+\boxed{\frac{1}{n}}}{1+\boxed{\frac{2}{n}}} = \frac{2}{3} < 1.$$

Denne konvergerer.

c) $\sum_{n=2}^{\infty} \frac{2n^2}{n^2-1}$

Divergenstesten:

$$\lim_{n\to\infty} \frac{2n^2}{n^2-1} = \lim_{n\to\infty} \frac{2}{1-\boxed{1/n^2}} = 2 \neq 0.$$

Divergenstesten sier at dette divergerer.

8.4.3

a) $\sum_{n=0}^{\infty} \frac{\boxed{n^{999}}}{\boxed{n!}} = 0 + 1 + \frac{2^{999}}{2} +$

$$\lim_{n\to\infty} \frac{n^{999}}{n!} = 0$$ Vanskelig å regne ut, la oss hoppe rett til forholdstesten.

Forholdstesten

$$\lim_{n\to\infty} \frac{\frac{(n+1)^{999}}{(n+1)!}}{\frac{n^{999}}{n!}} = \lim_{n\to\infty} \frac{(n+1)^{999}}{(n+1)!} \cdot \frac{n!}{n^{999}}$$

$$= \lim_{n\to\infty} \frac{n!}{(n+1)!} \cdot \left(\frac{n+1}{n}\right)^{999}$$

$$= \lim_{n\to\infty} \frac{\cancel{1\cdot 2\cdot 3\cdot 4\cdot \ldots \cdot n}}{\cancel{1\cdot 2\cdot 3\cdot 4\cdot \ldots \cdot n}\cdot(n+1)} \cdot \left(1+\frac{1}{n}\right)^{999}$$

$$= \lim_{n\to\infty} \frac{1}{n+1} \cdot \boxed{\left(1+\boxed{\frac{1}{n}}\right)}^{999} = 0 < 1$$

Rekka konvergerer.

b) $\sum_{n=2}^{\infty} \frac{1}{n\ln(n)}$

Integraltesten:

Variabelskifte, $u = \ln x$

$\frac{du}{dx} = \frac{1}{x}$

$du = \frac{1}{x} dx$

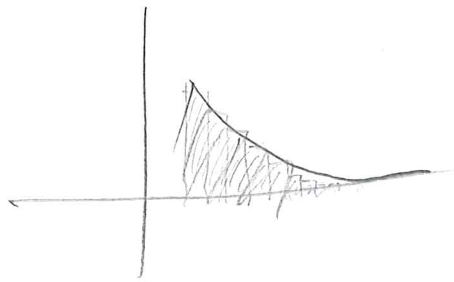
$$\int_2^{\infty} \frac{1}{x\ln x} dx$$

$$= \int_{"2"}^{"\infty"} \frac{1}{u} du = \left[\ln u\right]_{"2"}^{"\infty"} = \left[\ln(\ln x)\right]_2^{\infty}$$

$$= \ln(\ln \infty) - \ln(\ln(2))$$

$$= \infty - \ln(\ln(2)) = \infty.$$

c) også, må bruke integraltesten.



8.4.4

a) $\sum_{n=1}^{\infty} \cos\left(\frac{1}{n^2}\right)$

Divergenstesten

$$\lim_{n\to\infty} \cos\left(\frac{1}{n^2}\right) = \cos(0) = 1 \neq 0$$

Så rekka divergerer.

b) Integraltest (Variabelskifte, $u = \ln x$)

c)

$$\sum_{n=0}^{\infty} \frac{n!(n+1)!}{(2n+1)!}$$

Prøver forholdstesten

$2\boxed{(n+1)} + 1 = 2n + \boxed{2+1}$

$$\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n\to\infty} \frac{\frac{(n+1)!\cdot(n+2)!}{(2n+3)!}}{\frac{n!\,(n+1)!}{(2n+1)!}} = \lim_{n\to\infty} \frac{\cancel{(n+1)!}(n+2)!}{(2n+3)!} \cdot \frac{(2n+1)!}{n!\cancel{(n+1)!}}$$

$$= \lim_{n\to\infty} \frac{\cancel{n!}\cdot(n+1)\cdot(n+2)\cdot\cancel{(2n+1)!}}{\cancel{n!}\;\cancel{(2n+1)!}\cdot(2n+2)\cdot(2n+3)} = \lim_{n\to\infty} \frac{(n+1)(n+2)}{(2n+2)(2n+3)}$$

$$= \lim_{n\to\infty} \frac{\left(1+\boxed{\frac{1}{n}}\right)\left(1+\boxed{\frac{2}{n}}\right)}{\left(2+\boxed{\frac{2}{n}}\right)\left(2+\boxed{\frac{3}{n}}\right)} = \frac{1\cdot 1}{2\cdot 2} = \frac{1}{4} < 1$$

Rekka konvergerer.

8.7.1

$$\boxed{\sum_{n=1}^{\infty} \frac{3n-1}{n^4+2}}$$

Ser at dette er ca $\frac{n}{n^4} = \boxed{\frac{1}{n^3}}$

Grensesammenlikningstesten:

$$\lim_{n\to\infty} \frac{\frac{3n-1}{n^4+2}}{1/n^3} = \lim_{n\to\infty} \frac{3n-1}{n^4+2}\cdot\frac{n^3}{1} = \lim_{n\to\infty} \frac{3n^4-n^3}{n^4+2}$$

$$= \lim_{n\to\infty} \frac{3-\boxed{\frac{1}{n}}}{1+\boxed{\frac{2}{n^4}}} = \frac{3}{1} = 3$$

Vi kan si at

$$\sum_{n=1}^{\infty} \frac{3n-1}{n^4+2}$$ konvergerer hvis $\boxed{\sum_{n=1}^{\infty} \frac{1}{n^3}}$ konvergerer

Denne konvergerer fordi $p=3>1$.

b) $$\sum_{n=1}^{\infty} \frac{5+6n^2}{2n+3n^2}$$ likner på $\frac{\cancel{n^2}}{\cancel{n^2}} = 1$ $\quad \sum 1$ divergerer.

Divergenstesten : $\lim_{n \to \infty} \frac{5+6n^2}{2n+3n^2} = \lim_{n \to \infty} \frac{\boxed{\frac{5}{n^2}}+6}{\boxed{\frac{2}{n}}+3} = 2 \neq 0.$

Divergerer.

c) $$\sum_{n=1}^{\infty} \frac{\sqrt{n}}{n^2+1}$$ likner på $\frac{\sqrt{n}}{n^2} = \frac{n^{1/2}}{n^2} = \frac{1}{n^{1.5}}$. Vet at $\sum \frac{1}{n^{1.5}}$ konvergerer, $1.5 > 1$.

$$\lim_{n \to \infty} \frac{\frac{\sqrt{n}}{n^2+1}}{1/n^{1.5}} = \lim_{n \to \infty} \frac{\sqrt{n}}{n^2+1} \cdot \frac{n^{1.5}}{1} = \lim_{n \to \infty} \frac{n^{0.5+1.5}}{n^2+1} = \lim_{n \to \infty} \frac{n^2}{n^2+1}$$

$$= \lim_{n \to \infty} \frac{1}{1+\frac{1}{n^2}} = \boxed{1}$$ og siden $0 < 1 < \infty$,

vil vår rekke konvergere.

7.8.1 Kalkulus:

a) $$\sum_{n=1}^{\infty} \frac{7n^2+3}{4n^3-2} \approx \frac{n^2}{n^3} = \frac{1}{n}, \quad \sum \frac{1}{n} \text{ divergerer.}$$

Grensesammenlikner:

$$\lim_{n\to\infty} \frac{\frac{7n^2+3}{4n^3-2}}{1/n} = \lim_{n\to\infty} \frac{7n^2+3}{4n^3-2}\cdot\frac{n}{1} = \lim_{n\to\infty} \frac{7n^3+3n}{4n^3-2}$$

$$= \lim_{n\to\infty} \frac{7+\boxed{3/n^2}}{4-\boxed{2/n^3}} = \frac{7}{4}$$, og siden $0 < \frac{7}{4} < \infty$,

vil også vår rekke divergere.

c) $$\sum_{n=0}^{\infty} \frac{\arctan n}{n^2+1}$$

$$\frac{\arctan n}{n^2+1} \leq \frac{\pi/2}{n^2}$$

Så $$\boxed{\sum \frac{\arctan n}{n^2+1}} \leq \sum \frac{\pi/2}{n^2} = \frac{\pi}{2}\boxed{\sum \frac{1}{n^2}}$$

Konvergerer.

Må også konvergere.