12.7.2

a) Diagonalisér $A = \begin{pmatrix} -5 & 6 \\ 4 & 5 \end{pmatrix}$

$$\begin{vmatrix} -5-\lambda & 6 \\ 4 & 5-\lambda \end{vmatrix} = (-5-\lambda)(5-\lambda) - 24$$

$$= \lambda^2 - 25 - 24 = \lambda^2 - 49 = 0$$

$\lambda = 7$ og $\lambda = -7$.

Egenvektor til $\lambda = 7$ $\qquad (A - \lambda I)\vec{v} = \vec{0}$

$$\begin{pmatrix} -5-\lambda & 6 & 0 \\ 4 & 5-\lambda & 0 \end{pmatrix} = \begin{pmatrix} -12 & 6 & 0 \\ 4 & -2 & 0 \end{pmatrix} \overset{\frac{1}{2}R_2}{\sim} \begin{pmatrix} -12 & 6 & 0 \\ 2 & -1 & 0 \end{pmatrix}$$

$$\overset{R_1 + 6 \cdot R_2}{\sim} \begin{pmatrix} 0 & 0 & 0 \\ 2 & -1 & 0 \end{pmatrix} \Rightarrow \begin{array}{l} 2x - y = 0 \\ y = 2x \end{array}$$

Egenvektor $\begin{pmatrix} 1 \\ 2 \end{pmatrix}$.

Egenvektor til $\lambda = -7$ $\qquad \begin{pmatrix} 2 & 6 & 0 \\ 4 & 12 & 0 \end{pmatrix}$

$$\begin{pmatrix} 2 & 6 & 0 \\ 4 & 12 & 0 \end{pmatrix} \overset{\frac{1}{2}R_1}{\sim} \begin{pmatrix} 1 & 3 & 0 \\ 4 & 12 & 0 \end{pmatrix} \overset{R_2-4R_1}{\sim} \begin{pmatrix} 1 & 3 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$\Rightarrow \quad x + 3y = 0 \qquad x = -3y$

Egenvektor $\begin{pmatrix} -3 \\ 1 \end{pmatrix}$.

Diagonalisér $\begin{pmatrix} -5 & 6 \\ 4 & 5 \end{pmatrix} = \begin{pmatrix} 1 & -3 \\ 2 & 1 \end{pmatrix} \begin{pmatrix} 7 & 0 \\ 0 & -7 \end{pmatrix} \begin{pmatrix} 1 & -3 \\ 2 & 1 \end{pmatrix}^{-1}$

Gir $P = \begin{pmatrix} 1 & -3 \\ 2 & 1 \end{pmatrix}$, $D = \begin{pmatrix} 7 & 0 \\ 0 & -7 \end{pmatrix}$

$A = P \cdot D \cdot P^{-1}$

b) Finn den generelle løsningen til

$$x'(t) = -5x(t) + 6y(t)$$
$$y'(t) = 4x(t) + 5y(t)$$

Skriver om som matrisesystem

$$\begin{pmatrix} x(t) \\ y(t) \end{pmatrix}' = \begin{pmatrix} -5 & 6 \\ 4 & 5 \end{pmatrix} \begin{pmatrix} x(t) \\ y(t) \end{pmatrix}$$

Har egenverdier $\lambda = 7 \qquad \lambda = -7$

egenvektorer $\begin{pmatrix} 1 \\ 2 \end{pmatrix} \qquad \begin{pmatrix} -3 \\ 1 \end{pmatrix}$

Generell løsning gir

$$\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = C \cdot e^{7t} \begin{pmatrix} 1 \\ 2 \end{pmatrix} + D \cdot e^{-7t} \begin{pmatrix} -3 \\ 1 \end{pmatrix}$$

$$= \begin{pmatrix} Ce^{7t} \\ 2Ce^{7t} \end{pmatrix} + \begin{pmatrix} -3De^{-7t} \\ De^{-7t} \end{pmatrix}$$

$$\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = \begin{pmatrix} Ce^{7t} - 3De^{-7t} \\ 2Ce^{7t} + De^{-7t} \end{pmatrix}$$

$$x(t) = Ce^{7t} - 3De^{-7t}$$
$$y(t) = 2Ce^{7t} + De^{-7t}$$

c) Finn det stasjonære punktet til

$$x'(t) = -5x(t) + 6y(t) - 1$$
$$y'(t) = 4x(t) + 5y(t) - 9$$

„Løs den inhomogene diffligningen"

Matriseform

$$\begin{pmatrix} x(t) \\ y(t) \end{pmatrix}' = \begin{pmatrix} -5 & 6 \\ 4 & 5 \end{pmatrix} \begin{pmatrix} x(t) \\ y(t) \end{pmatrix} + \begin{pmatrix} -1 \\ -9 \end{pmatrix}$$

Løsning:

La $\vec{x}_c$ være løsning av $\vec{x}_c' = A\vec{x}_c$.

Da er $\vec{x} = \vec{x}_c - A^{-1}\vec{b}$ løsningen av

$$\vec{x}' = A\vec{x} + \vec{b}$$

Vi må regne ut $A^{-1}\vec{b}$ er.

$$A = \begin{pmatrix} -5 & 6 \\ 4 & 5 \end{pmatrix} \qquad \vec{b} = \begin{pmatrix} -1 \\ -9 \end{pmatrix} \qquad |A| = -49$$

$$A^{-1} = \frac{1}{-49} \cdot \begin{pmatrix} 5 & -6 \\ -4 & -5 \end{pmatrix} = \begin{pmatrix} \frac{-5}{49} & \frac{6}{49} \\ \frac{4}{49} & \frac{5}{49} \end{pmatrix}$$

$$\begin{pmatrix} -5 & 6 & 1 & 0 \\ 4 & 5 & 0 & 1 \end{pmatrix} \overset{R_1+R_2}{\sim} \begin{pmatrix} -1 & 11 & 1 & 1 \\ 4 & 5 & 0 & 1 \end{pmatrix} \overset{-1 \cdot R_1}{\sim} \begin{pmatrix} 1 & -11 & -1 & -1 \\ 4 & 5 & 0 & 1 \end{pmatrix}$$

$$\overset{R_2-4R_1}{\sim} \begin{pmatrix} 1 & -11 & -1 & -1 \\ 0 & 49 & 4 & 5 \end{pmatrix} \overset{\frac{1}{49} \cdot R_2}{\sim} \begin{pmatrix} 1 & -11 & -1 & -1 \\ 0 & 1 & \frac{4}{49} & \frac{5}{49} \end{pmatrix}$$

$$-\frac{49}{49} + \frac{44}{49} = -\frac{5}{49}$$

$$-\frac{49}{49} + \frac{55}{49} = \frac{6}{49}$$

$$\overset{R_1+11 \cdot R_2}{\sim} \begin{pmatrix} 1 & 6 & -\frac{5}{49} & \frac{6}{49} \\ 0 & 1 & \frac{4}{49} & \frac{5}{49} \end{pmatrix}$$

$$A^{-1}\vec{b} = \begin{pmatrix} -\frac{5}{49} & \frac{6}{49} \\ \frac{4}{49} & \frac{5}{49} \end{pmatrix} \begin{pmatrix} -1 \\ -9 \end{pmatrix} = \begin{pmatrix} \frac{5}{49} - \frac{54}{49} \\ -\frac{4}{49} - \frac{45}{49} \end{pmatrix} = \begin{pmatrix} -1 \\ -1 \end{pmatrix}$$

$$\vec{x} = \vec{x}_c - A^{-1}\vec{b} = \vec{x}_c - \begin{pmatrix} -1 \\ -1 \end{pmatrix} = \vec{x}_c + \boxed{\begin{pmatrix} 1 \\ 1 \end{pmatrix}}$$

Det stasjonære punktet er $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$, $x=1, y=1$.

Løsningen av differensiallikningen blir

$$\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = \begin{pmatrix} Ce^{7t} - 3De^{-7t} \\ 2Ce^{7t} + De^{-7t} \end{pmatrix} + \begin{pmatrix} 1 \\ 1 \end{pmatrix} \leftarrow -A^{-1}\vec{b}$$

$$x(t) = Ce^{7t} - 3De^{-7t} + 1$$
$$y(t) = 2Ce^{7t} + De^{-7t} + 1$$

d) Finn løsningen av differensiallikningen fra c) som tilfredsstiller $x(0) = 0$ og $y(0) = 0$.

$$0 = Ce^{7\cdot 0} - 3De^{-7\cdot 0} + 1$$
$$0 = 2Ce^{7\cdot 0} + De^{-7\cdot 0} + 1$$

$$0 = C - 3D + 1 \qquad C - 3D = -1$$
$$0 = 2C + D + 1 \qquad 2C + D = -1$$

$$\begin{pmatrix} 1 & -3 & -1 \\ 2 & 1 & -1 \end{pmatrix}$$

$$\begin{pmatrix} 1 & -3 & -1 \\ 2 & 1 & -1 \end{pmatrix} \overset{R_2-2R_1}{\sim} \begin{pmatrix} 1 & -3 & -1 \\ 0 & 7 & 1 \end{pmatrix} \overset{\frac{1}{7}R_2}{\sim} \begin{pmatrix} 1 & -3 & -1 \\ 0 & 1 & 1/7 \end{pmatrix}$$

$$\overset{R_1+3R_2}{\sim} \begin{pmatrix} 1 & 0 & -4/7 \\ 0 & 1 & 1/7 \end{pmatrix} \qquad C = -4/7 \qquad D = 1/7$$

Løsningen:

$$x(t) = -\frac{4}{7}e^{7t} - \frac{3}{7}e^{-7t} + 1$$

$$y(t) = -\frac{8}{7}e^{7t} + \frac{1}{7}e^{-7t} + 1$$

12.7.31

a) $y' = 0y + 3z$

$z' = -3y + 0z$

Matrise:

$$\begin{pmatrix} y \\ z \end{pmatrix}' = \begin{pmatrix} 0 & 3 \\ -3 & 0 \end{pmatrix} \begin{pmatrix} y \\ z \end{pmatrix}$$

$\uparrow$

Finn egenverdier for A

$$\begin{vmatrix} -\lambda & 3 \\ -3 & -\lambda \end{vmatrix} = \lambda^2 + 9 = 0$$

$\lambda^2 = -9$

$\lambda = \pm\sqrt{-9}$

$= \pm 3i$

Egenvektor for $\lambda = 3i$

$$\left(\begin{array}{cc|c} -3i & 3 & 0 \\ -3 & -3i & 0 \end{array}\right) \underset{-\frac{1}{3}R_2}{\overset{R_1 - iR_2}{\sim}} \left(\begin{array}{cc|c} 0 & 0 & 0 \\ 1 & i & 0 \end{array}\right)$$

$x + iy = 0 \qquad x = -iy$

Egenvektor $\begin{pmatrix} -i \\ 1 \end{pmatrix}$

Egenvektor for $\lambda = -3i$ blir $\overline{\begin{pmatrix} -i \\ 1 \end{pmatrix}} = \begin{pmatrix} -\bar{i} \\ \bar{1} \end{pmatrix} = \begin{pmatrix} i \\ 1 \end{pmatrix}$

For komplekse egenverdie / vektorer:

$$\begin{pmatrix} y \\ z \end{pmatrix} = C \cdot Re\left(e^{\lambda_1 t}\vec{v_1}\right) + D \cdot Im\left(e^{\lambda_1 t}\vec{v_1}\right)$$

Se på $e^{\lambda_1 t}\vec{v_1} = e^{3it} \cdot \begin{pmatrix} -i \\ 1 \end{pmatrix} = \begin{pmatrix} \\ \end{pmatrix} + i \begin{pmatrix} \\ \end{pmatrix}$

$$e^{3ti} = \cos 3t + i \sin 3t$$

$$e^{\lambda_1 t}\vec{v_1} = (\cos 3t + i \sin 3t)\begin{pmatrix} -i \\ 1 \end{pmatrix}$$

$$= \begin{pmatrix} -i(\cos 3t + i \sin 3t) \\ 1 \cdot (\cos 3t + i \sin 3t) \end{pmatrix} = \begin{pmatrix} \sin 3t - i \cos 3t \\ \cos 3t + i \sin 3t \end{pmatrix}$$

$$= \begin{pmatrix} \sin 3t \\ \cos 3t \end{pmatrix} + i \begin{pmatrix} -\cos 3t \\ \sin 3t \end{pmatrix}$$

$$Re\left(e^{\lambda_1 t}\vec{v_1}\right) = \begin{pmatrix} \sin 3t \\ \cos 3t \end{pmatrix} \quad Im\left(e^{\lambda_1 t}\vec{v_1}\right) = \begin{pmatrix} -\cos 3t \\ \sin 3t \end{pmatrix}$$

$$\begin{pmatrix} y \\ z \end{pmatrix} = C \cdot \begin{pmatrix} \sin 3t \\ \cos 3t \end{pmatrix} + D \begin{pmatrix} -\cos 3t \\ \sin 3t \end{pmatrix}$$

$$y(t) = C \sin 3t - D \cos 3t$$
$$z(t) = C \cos 3t + D \sin 3t$$

b) $y' = y + 2z$

$z' = -2y + z$

$$\begin{pmatrix} y \\ z \end{pmatrix}' = \begin{pmatrix} 1 & 2 \\ -2 & 1 \end{pmatrix} \begin{pmatrix} y \\ z \end{pmatrix}$$

$$\begin{vmatrix} 1-\lambda & 2 \\ -2 & 1-\lambda \end{vmatrix} = (1-\lambda)^2 + 4 = \lambda^2 - 2\lambda + 1 + 4$$

$$= \lambda^2 - 2\lambda + 5 = 0$$

$$\lambda = \frac{2 \pm \sqrt{4-20}}{2} = \frac{2 \pm 4i}{2} = 1 \pm 2i$$

Egenvektor for $\lambda = 1 + 2i$

$$\begin{pmatrix} -2i & 2 & 0 \\ -2 & -2i & 0 \end{pmatrix} \overset{R_1 - iR_2}{\sim} \begin{pmatrix} 0 & 0 & 0 \\ -2 & -2i & 0 \end{pmatrix} \overset{-\frac{1}{2}R_2}{\sim} \begin{pmatrix} 0 & 0 & 0 \\ 1 & i & 0 \end{pmatrix}$$

~~Eg~~ $\Rightarrow$ $x + iy = 0$ $\quad x = -iy$

Egenvektor $\begin{pmatrix} -i \\ 1 \end{pmatrix}$

Får egenvektor for $1-2i$ blir $\begin{pmatrix} i \\ 1 \end{pmatrix}$

Må igjen finne realdel og imaginær del av

$e^{(1+2i)t} \begin{pmatrix} -i \\ 1 \end{pmatrix}$. Ser på denne.

$$e^{(1+2i)t}\begin{pmatrix}-i\\1\end{pmatrix}$$

$$e^{(1+2i)t}=e^{t+2it}=e^{t}\cdot \boxed{e^{2ti}}$$

$$=e^{t}(\cos 2t+i\sin 2t)$$

$$e^{(1+2i)t}\begin{pmatrix}-i\\1\end{pmatrix}=e^{t}(\cos 2t+i\sin 2t)\begin{pmatrix}-i\\1\end{pmatrix}$$

$$=\begin{pmatrix}-ie^{t}(\cos 2t+i\sin 2t)\\e^{t}(\cos 2t+i\sin 2t)\end{pmatrix}$$

$$=\begin{pmatrix}e^{t}\sin 2t-ie^{t}\cos 2t\\e^{t}\cos 2t+ie^{t}\sin 2t\end{pmatrix}$$

$$=\begin{pmatrix}e^{t}\sin 2t\\e^{t}\cos 2t\end{pmatrix}+i\begin{pmatrix}-e^{t}\cos 2t\\e^{t}\sin 2t\end{pmatrix}$$

$$\begin{pmatrix}y\\z\end{pmatrix}=C\cdot\begin{pmatrix}e^{t}\sin 2t\\e^{t}\cos 2t\end{pmatrix}+D\begin{pmatrix}-e^{t}\cos 2t\\e^{t}\sin 2t\end{pmatrix}$$

$$y=Ce^{t}\sin 2t-De^{t}\cos 2t$$

$$z=Ce^{t}\cos 2t+De^{t}\sin 2t$$

c) $y' = 3y - z$

$z' = 4y + 3z$

$$\begin{pmatrix} y \\ z \end{pmatrix}' = \begin{pmatrix} 3 & -1 \\ 4 & 3 \end{pmatrix} \begin{pmatrix} y \\ z \end{pmatrix}$$

$$\begin{vmatrix} 3-\lambda & -1 \\ 4 & 3-\lambda \end{vmatrix} = (3-\lambda)^2 + 4 = \lambda^2 - 6\lambda + 9 + 4$$

$$= \lambda^2 - 6\lambda + 13 = 0$$

$$\lambda = \frac{6 \pm \sqrt{36-52}}{2} = \frac{6 \pm 4i}{2} = 3 \pm 2i$$

Egenvektor til $3+2i$

$$\begin{pmatrix} -2i & -1 & 0 \\ 4 & -2i & 0 \end{pmatrix} \underset{R_1 \cdot (-1)}{\overset{R_2 + 2iR_1}{\sim}} \begin{pmatrix} 2i & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$2ix + y = 0$ $\quad y = -2ix$ $\quad$ Velg $x = 1$

Egenvektor $\begin{pmatrix} 1 \\ -2i \end{pmatrix}$

Egenvektor til $3-2i$ blir $\begin{pmatrix} 1 \\ 2i \end{pmatrix}$

Se på $e^{(3+2i)t} \begin{pmatrix} 1 \\ -2i \end{pmatrix}$

$$e^{(3+2i)t} = e^{3t + 2it} = e^{3t} \cdot e^{2ti}$$

$$= e^{3t}(\cos 2t + i \sin 2t)$$

$$e^{3t}(\cos 2t + i\sin 2t)\begin{pmatrix}1\\-2i\end{pmatrix}$$

$$= \begin{pmatrix} e^{3t}\cos 2t + ie^{3t}\sin 2t \\ -2ie^{3t}(\cos 2t + i\sin 2t) \end{pmatrix}$$

$$= \begin{pmatrix} e^{3t}\cos 2t + ie^{3t}\sin 2t \\ 2e^{3t}\sin 2t - i2e^{3t}\cos 2t \end{pmatrix}$$

$$= \begin{pmatrix} e^{3t}\cos 2t \\ 2e^{3t}\sin 2t \end{pmatrix} + i\begin{pmatrix} e^{3t}\sin 2t \\ -2e^{3t}\cos 2t \end{pmatrix}$$

Generelle løsning blir

$$\begin{pmatrix} y \\ z \end{pmatrix} = C\cdot\begin{pmatrix} e^{3t}\cos 2t \\ 2e^{3t}\sin 2t \end{pmatrix} + D\cdot\begin{pmatrix} e^{3t}\sin 2t \\ -2e^{3t}\cos 2t \end{pmatrix}$$

$$y(t) = Ce^{3t}\cos 2t + De^{3t}\sin 2t$$
$$z(t) = 2Ce^{3t}\sin 2t - 2De^{3t}\cos 2t$$

Vi bruker: $e^{a+ib} = e^a \cdot e^{ib}$

$$e^{ib} = \cos b + i\sin b$$

12.7.4

$$y' = 2y - 4z$$
$$z' = 2y - 2z$$

$$\begin{pmatrix} y \\ z \end{pmatrix}' = \begin{pmatrix} 2 & -4 \\ 2 & -2 \end{pmatrix} \begin{pmatrix} y \\ z \end{pmatrix}$$

Egenverdier:

$$\begin{vmatrix} 2-\lambda & -4 \\ 2 & -2-\lambda \end{vmatrix} = (2-\lambda)(-2-\lambda) + 8 = 0$$

$$= \lambda^2 - 4 + 8 = 0$$

$$\Rightarrow \lambda^2 + 4 = 0 \qquad \lambda^2 = -4$$
$$\lambda = \pm 2i$$

Egenvektor: for $\lambda = 2i$

$$\begin{pmatrix} 2-2i & -4 & 0 \\ 2 & -2-2i & 0 \end{pmatrix} \underset{\frac{1}{2}R_2}{\overset{R_1 - \cancel{a}R_2}{\sim}} \begin{pmatrix} 0 & 0 & 0 \\ 1 & -1-i & 0 \end{pmatrix}$$

$$x + (-1-i)y = 0 \qquad x = (1+i)y \qquad y = 1$$

Egenvektor $\begin{pmatrix} 1+i \\ 1 \end{pmatrix}$

Må se på $e^{2it} \begin{pmatrix} 1+i \\ 1 \end{pmatrix}$

$$e^{2it} = \cos 2t + i \sin 2t$$

$$\left(\cos 2t + i\sin 2t\right)\begin{pmatrix} 1+i \\ 1 \end{pmatrix}$$

$$= \begin{pmatrix} (1+i)(\cos 2t + i\sin 2t) \\ \cos 2t + i\sin 2t \end{pmatrix}$$

$$= \begin{pmatrix} \cos 2t + i\sin 2t + i\cos 2t - \sin 2t \\ \cos 2t + i\sin 2t \end{pmatrix}$$

$$= \begin{pmatrix} \cos 2t - \sin 2t + i(\sin 2t + \cos 2t) \\ \cos 2t \quad + i\sin 2t \end{pmatrix}$$

$$= \begin{pmatrix} \cos 2t - \sin 2t \\ \cos 2t \end{pmatrix} + i\begin{pmatrix} \sin 2t + \cos 2t \\ \sin 2t \end{pmatrix}$$

$$\begin{pmatrix} y \\ z \end{pmatrix} = C\cdot\begin{pmatrix} \cos 2t - \sin 2t \\ \cos 2t \end{pmatrix} + D\begin{pmatrix} \sin 2t + \cos 2t \\ \sin 2t \end{pmatrix}$$

$$y(t) = C\cdot(\cos 2t - \sin 2t) + D(\sin 2t + \cos 2t)$$

$$z(t) = C\cdot\cos 2t \quad + D\cdot\sin 2t.$$

$$y(t) = (C+D)\cos 2t + (D-C)\sin 2t$$

$$z(t) = C \cdot \cos 2t + D \cdot \sin 2t$$