10.1.4

a) Tre matriser

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$$

har alle <u>samme</u> karakteristiske polynom, og bare én egenverdi.
Hva er dimensjonen til egenrommene.

$$|A - \lambda I| = \left| \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix} - \lambda \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \right|$$

$$= \begin{vmatrix} 1-\lambda & 1 & 0 \\ 0 & 1-\lambda & 1 \\ 0 & 0 & 1-\lambda \end{vmatrix} = (1-\lambda) \begin{vmatrix} 1-\lambda & 1 \\ 0 & 1-\lambda \end{vmatrix} = (1-\lambda)^3 = 0$$

Får $\lambda = 1$.

Egenvektor til $\lambda = 1$

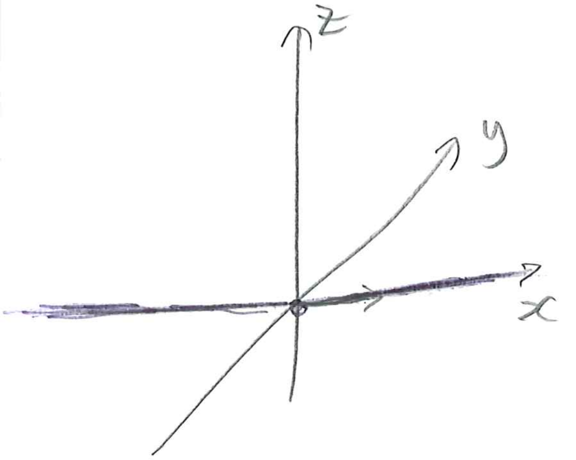
$$\begin{pmatrix} 1-\lambda & 1 & 0 & 0 \\ 0 & 1-\lambda & 1 & 0 \\ 0 & 0 & 1-\lambda & 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$y = 0 \qquad z = 0 \qquad 0 = 0$

Egenvektor $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x \\ 0 \\ 0 \end{pmatrix} = x \cdot \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$

Alle mulige egenvektorer:

Rett linje, én-dimensjonal.



Neste:

$$\left|\begin{pmatrix}1&1&0\\0&1&0\\0&0&1\end{pmatrix} - \lambda\begin{pmatrix}1&0&0\\0&1&0\\0&0&1\end{pmatrix}\right| = \begin{vmatrix}1-\lambda&1&0\\0&1-\lambda&0\\0&0&1-\lambda\end{vmatrix} = (1-\lambda)^3$$

$\lambda = 1$ egenvektor

To frie variable

$$\begin{pmatrix}0&1&0&0\\0&0&0&0\\0&0&0&0\end{pmatrix}$$

$y = 0$

$0 = 0$

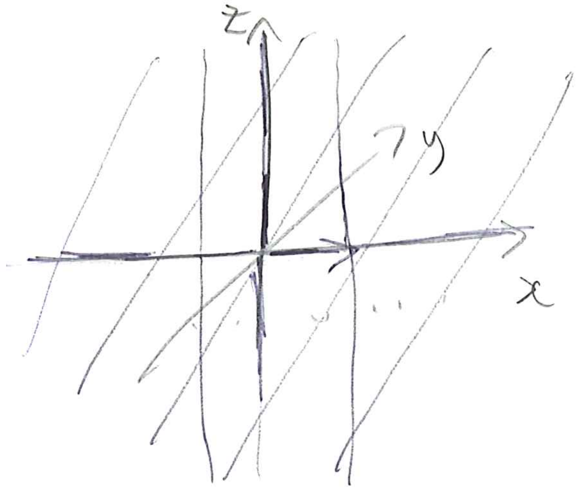
$0 = 0$

$$\begin{pmatrix}x\\y\\z\end{pmatrix} = \begin{pmatrix}x\\0\\z\end{pmatrix} = \begin{pmatrix}x\\0\\0\end{pmatrix} + \begin{pmatrix}0\\0\\z\end{pmatrix} = x\cdot\begin{pmatrix}1\\0\\0\end{pmatrix} + z\begin{pmatrix}0\\0\\1\end{pmatrix}$$

$$(A-\lambda I)\vec{x} = \vec{0}$$

$$A\vec{x} = \lambda\vec{x}$$

Alle mulige egenvektorer:



$$\left| \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} - \lambda \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \right| = (1-\lambda)^3$$

$\lambda = 1$

$$\begin{pmatrix} 1-\lambda & 0 & 0 & 0 \\ 0 & 1-\lambda & 0 & 0 \\ 0 & 0 & 1-\lambda & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

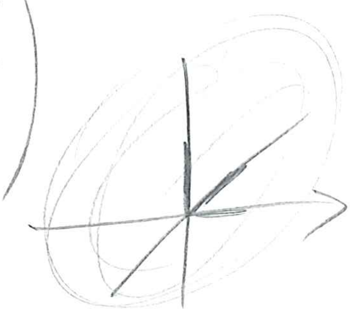
$0 = 0$

$0 = 0$

$0 = 0$

$$I \cdot \vec{x} = 1 \cdot \vec{x}$$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = x \cdot \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + y \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} + z \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$



b) I 4D, finn fire 4x4-matriser, med samme karakteristiske polynom, én egenverdi, og 1, 2, 3, og 4-dimensjonale egenrom.

$$A_1 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = A_2,$$

$$A_3 = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix} = A_4$$

Karakteristisk polynom $(1-\lambda)^4$, $\lambda = 1$

Egenvektorer:

$$A_1: \left(\begin{array}{cccc|c} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right) \qquad \begin{array}{c} 0=0 \\ 0=0 \\ 0=0 \\ 0=0 \end{array}$$

$$\begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix} = x\begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} + y\begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} + z\begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} + w\begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}$$

4 dimensjonalt egenrom

A_2: $(A_2 - \lambda I)\vec{x} = \vec{0}$

$$\begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \Rightarrow \begin{matrix} y = 0 \\ 0 = 0 \\ 0 = 0 \\ 0 = 0 \end{matrix}$$

$$\begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix} = \begin{pmatrix} x \\ 0 \\ z \\ w \end{pmatrix} = x\begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} + z\begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} + w\begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}$$

3-dimensjonalt egenrom.

A_3:

$$\begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \Rightarrow \begin{matrix} y = 0 \\ z = 0 \\ 0 = 0 \\ 0 = 0 \end{matrix}$$

$$\begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix} = \begin{pmatrix} x \\ 0 \\ 0 \\ w \end{pmatrix} = x\begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} + w\begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}$$

2-dimensjonalt egenrom.

A_4: $(A_4 - \lambda I)\vec{x} = \vec{0}$

$$\begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix} - \begin{pmatrix} \lambda & 0 & 0 & 0 \\ 0 & \lambda & 0 & 0 \\ 0 & 0 & \lambda & 0 \\ 0 & 0 & 0 & \lambda \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\left(\begin{array}{cccc|c} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right) \qquad \begin{array}{l} y = 0 \\ z = 0 \\ w = 0 \\ 0 = 0 \end{array}$$

$$\begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix} = \begin{pmatrix} x \\ 0 \\ 0 \\ 0 \end{pmatrix} = x \cdot \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

En-dimensionalt egenrom.

10.1.3

Finn egenverdi og egenvektor til

a) $\begin{pmatrix} 4 & 1 & 1 \\ 3 & 2 & -1 \\ 0 & 0 & 3 \end{pmatrix} = A$

$|A - \lambda I| = 0 \qquad \begin{vmatrix} 4-\lambda & 1 & 1 \\ 3 & 2-\lambda & -1 \\ 0 & 0 & 3-\lambda \end{vmatrix} = 0$

$(3-\lambda)\begin{vmatrix} 4-\lambda & 1 \\ 3 & 2-\lambda \end{vmatrix} = (3-\lambda)\left((4-\lambda)(2-\lambda) - 1\cdot 3\right)$

$= (3-\lambda)(\lambda^2 - 6\lambda + 5) = 0$

$3 - \lambda = 0 \Rightarrow \lambda = 3$

$\lambda^2 - 6\lambda + 5 = 0 \Rightarrow \lambda = 1 \vee \lambda = 5$

$\lambda = 1$

$\begin{pmatrix} 3 & 1 & 1 & 0 \\ 3 & 1 & -1 & 0 \\ 0 & 0 & 2 & 0 \end{pmatrix} \underset{\sim}{R_2 - R_1} \begin{pmatrix} 3 & 1 & 1 & 0 \\ 0 & 0 & -2 & 0 \\ 0 & 0 & 2 & 0 \end{pmatrix}$

$\begin{aligned} 3x + y + z &= 0 \\ -2z &= 0 \\ 2z &= 0 \end{aligned} \Rightarrow \begin{aligned} 3x + y &= 0 \\ z &= 0 \end{aligned} \qquad y = -3x$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x \\ -3x \\ 0 \end{pmatrix} = x \cdot \begin{pmatrix} 1 \\ -3 \\ 0 \end{pmatrix}$$

$\lambda = 3$

$$\begin{pmatrix} 1 & 1 & 1 & 0 \\ 3 & -1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \overset{R_2 - 3 \cdot R_1}{\sim} \begin{pmatrix} 1 & 1 & 1 & 0 \\ 0 & -4 & -4 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\overset{-\frac{1}{4} R_2}{\sim} \begin{pmatrix} 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \overset{R_1 - R_2}{\sim} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$x = 0$

$y + z = 0 \qquad y = -z$

$0 = 0$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ -z \\ z \end{pmatrix} = z \begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix}$$

$\lambda = 5$

$$\begin{pmatrix} -1 & 1 & 1 & 0 \\ 3 & -3 & -1 & 0 \\ 0 & 0 & -2 & 0 \end{pmatrix} \overset{R_2+3R_1}{\sim} \begin{pmatrix} -1 & 1 & 1 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & -2 & 0 \end{pmatrix}$$

$$-x+y+z=0 \Rightarrow -x+y=0 \qquad x=y$$

$$\boxed{\begin{matrix} 2z=0 \\ -2z=0 \end{matrix}} \Rightarrow z=0$$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} y \\ y \\ 0 \end{pmatrix} = y \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$$

v)

$$\left| \begin{pmatrix} 3 & 3 & 3 \\ 3 & 3 & 3 \\ 3 & 3 & 3 \end{pmatrix} - \lambda \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \right| = 0$$

$$\begin{vmatrix} 3-\lambda & 3 & 3 \\ 3 & 3-\lambda & 3 \\ 3 & 3 & 3-\lambda \end{vmatrix} \overset{R_3-R_2}{\sim} \begin{vmatrix} 3-\lambda & 3 & 3 \\ 3 & 3-\lambda & 3 \\ 0 & +\lambda & -\lambda \end{vmatrix}$$

$$\overset{R_1-R_2}{\sim} \begin{vmatrix} -\lambda & \lambda & 0 \\ 3 & 3-\lambda & 3 \\ 0 & \lambda & -\lambda \end{vmatrix} \overset{R_1+\frac{\lambda}{3}R_2}{\sim} \begin{vmatrix} \ominus 0 & \lambda+\frac{\lambda}{3}(3-\lambda) & \lambda \\ \ominus 3 & 3-\lambda & 3 \\ \oplus 0 & \lambda & -\lambda \end{vmatrix}$$

$$-3 \begin{vmatrix} 2\lambda-\frac{\lambda^2}{3} & \lambda \\ \lambda & -\lambda \end{vmatrix} = -3\left(\left(2\lambda-\frac{\lambda^2}{3}\right)\cdot(-\lambda) - \lambda\cdot\lambda\right)$$

$$= -3\left(-2\lambda^2+\frac{\lambda^3}{3}-\lambda^2\right) = -\lambda^3+9\lambda^2=0$$

$$\lambda^2(9-\lambda)=0$$

$\lambda=0,\ \lambda=0,$ og
$\lambda=9.$

$$\lambda\cdot\lambda\cdot(9-\lambda)=0$$

$$\underset{\lambda=0}{\uparrow} \quad \underset{\lambda=0}{\uparrow} \quad \underset{\lambda=9}{\uparrow}$$

$\lambda = 9$

$$\begin{pmatrix} -6 & 3 & 3 & 0 \\ 3 & -6 & 3 & 0 \\ 3 & 3 & -6 & 0 \end{pmatrix} \underset{R_1+2R_2}{\overset{R_3-R_2}{\sim}} \begin{pmatrix} 0 & -9 & 9 & 0 \\ 3 & -6 & 3 & 0 \\ 0 & 9 & -9 & 0 \end{pmatrix}$$

$$\overset{R_3+R_1}{\sim} \begin{pmatrix} 0 & -9 & 9 & 0 \\ 3 & -6 & 3 & 0 \\ 6 & 0 & 0 & 0 \end{pmatrix} \underset{\frac{1}{3}R_2}{\overset{-\frac{1}{9}R_1}{\sim}} \begin{pmatrix} 0 & 1 & -1 & 0 \\ 1 & -2 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\overset{R_2+2R_1}{\sim} \begin{pmatrix} 0 & 1 & -1 & 0 \\ 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$y - z = 0$

$x \quad -z = 0$

$x = z$

$y = z$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} z \\ z \\ z \end{pmatrix} = z \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

$\underline{\lambda = 0}$

$$\begin{pmatrix} 3 & 3 & 3 & 0 \\ 3 & 3 & 3 & 0 \\ 3 & 3 & 3 & 0 \end{pmatrix} \underset{R_3 - R_1}{\overset{R_2 - R_1}{\sim}} \begin{pmatrix} 3 & 3 & 3 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\overset{\frac{1}{3}R_1}{\sim} \begin{pmatrix} 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$x + y + z = 0$$
$$x = -y - z$$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -y - z \\ y \\ z \end{pmatrix} = \begin{pmatrix} -y \\ y \\ 0 \end{pmatrix} + \begin{pmatrix} -z \\ 0 \\ z \end{pmatrix}$$

$$= y \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} + z \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$$

10.1.5

A er en kvadratisk matrise slik at summen av hver rad er konstant. Vis at $x = \begin{pmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{pmatrix}$ er en egenvektor.

Eks

$$A: \begin{pmatrix} 1 & -2 & 5 \\ 7 & -3 & 0 \\ 2 & -5 & 7 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 4 \\ 4 \\ 4 \end{pmatrix} = 4 \cdot \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

$$A \begin{pmatrix} x_{11} & x_{12} & x_{13} & \cdots & x_{1n} \\ x_{21} & x_{22} & x_{23} & \cdots & x_{2n} \\ \vdots & & & & \\ x_{n1} & \cdots & & & x_{nn} \end{pmatrix} \begin{pmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{pmatrix} = \begin{pmatrix} x_{11} + x_{12} + \cdots + x_{1n} \\ x_{21} + x_{22} + \cdots + x_{2n} \\ \vdots \\ x_{n1} + x_{n2} + \cdots + x_{nn} \end{pmatrix}$$

$$= \begin{pmatrix} c \\ c \\ c \\ \vdots \\ c \end{pmatrix} = c \cdot \begin{pmatrix} 1 \\ 1 \\ 1 \\ \vdots \\ 1 \end{pmatrix}$$