Diagonalisering

En matrise er diagonal dersom den kun har 0 utenom diagonalen

Eks = $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & -1 \end{pmatrix}$ $\begin{pmatrix} 3 & 0 \\ 0 & 5 \end{pmatrix}$

$\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$ Må være kvadratiske.

Lette å jobbe med:

$D = \begin{pmatrix} 3 & 0 \\ 0 & 5 \end{pmatrix}$ $D^3 = D \cdot D \cdot D = \begin{pmatrix} 3^3 & 0 \\ 0 & 5^3 \end{pmatrix} = \begin{pmatrix} 27 & 0 \\ 0 & 125 \end{pmatrix}$

$D = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & -1 \end{pmatrix}$ $D^{10} = \begin{pmatrix} 1^{10} & 0 & 0 \\ 0 & 2^{10} & 0 \\ 0 & 0 & (-1)^{10} \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1024 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

$A = \begin{pmatrix} 1 & 2 \\ 5 & 3 \end{pmatrix}$ $A^2 = A \cdot A \neq \begin{pmatrix} 1 & 4 \\ 25 & 9 \end{pmatrix}$

$\begin{pmatrix} 1 & 2 \\ 5 & 3 \end{pmatrix}\begin{pmatrix} 1 & 2 \\ 5 & 3 \end{pmatrix} = \begin{pmatrix} 11 & 8 \\ 20 & 19 \end{pmatrix}$

$D\vec{x} = \vec{b}$ $\quad D = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & -1 \end{pmatrix}$ $\quad \vec{b} = \begin{pmatrix} 1 \\ 10 \\ -2 \end{pmatrix}$

$$x = 1$$
$$2y = 10$$
$$-z = -2$$

Diagonalisering

A er en $n \times n$ matrise, A har egen verdier $\lambda_1, \lambda_2, \dots, \lambda_n$ med egenvektorer $\vec{v}_1, \vec{v}_2, \dots \vec{v}_n$

La $P = \begin{pmatrix} \vec{v}_1 & \vec{v}_2 & \dots & \vec{v}_n \end{pmatrix}$ og $D = \begin{pmatrix} \lambda_1 & & & 0 \\ & \lambda_2 & & \\ & & \ddots & \\ 0 & & & \lambda_n \end{pmatrix}$

Da vil $A \cdot P = P \cdot D$.

Fordi: $\vec{e}_i = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} \leftarrow$ plass i.

Holder å vise: $A \cdot P \cdot \vec{e}_i = P \cdot D \cdot \vec{e}_i$ for alle i

Fordi: $B \cdot e_i$ er kolonne i fra B

Eks $\begin{pmatrix} 3 & 5 & 2 \\ 2 & 1 & 0 \\ 0 & 2 & 7 \end{pmatrix}\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 5 \\ 1 \\ 2 \end{pmatrix}$

$$A \cdot P \cdot \vec{e_i} = A \cdot \vec{V_i} = \lambda_i \cdot \vec{V_i}$$

$$P \cdot D \cdot \vec{e_i} = P \cdot (\lambda_i \vec{e_i}) = \lambda_i P \vec{e_i} = \lambda_i \vec{V_i}$$

Bevist at $A \cdot P = P \cdot D$

Eks:

$A = \begin{pmatrix} 1 & 4 \\ 1 & 1 \end{pmatrix}$ $\quad \left| \begin{pmatrix} 1-\lambda & 4 \\ 1 & 1-\lambda \end{pmatrix} \right| = 0$

$(1-\lambda)(1-\lambda) - 4 = 0$

$\lambda^2 - 2\lambda - 3 = 0$ $\quad \lambda_1 = 3$ og $\lambda_2 = -1$

$\lambda_1 = 3$

$A \cdot \vec{v} = 3 \cdot \vec{v}$ $\quad \begin{pmatrix} 1 & 4 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 3x \\ 3y \end{pmatrix}$

$x + 4y = 3x$

$x + y = 3y$

$-2x + 4y = 0$

$x - 2y = 0$

$x = 2y$

Velger $y = 1$, så får $\vec{v_1} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$

$\lambda_2 = -1$

$\begin{pmatrix} 1 & 4 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -x \\ -y \end{pmatrix}$

$x + 4y = -x$

$x + y = -y$

$2x + 4y = 0$

$x + 2y = 0$

$x = -2y$

Velger $y = 1$, så får $\vec{v_2} = \begin{pmatrix} -2 \\ 1 \end{pmatrix}$

$$P = \begin{pmatrix} \vec{v}_1 & \vec{v}_2 \end{pmatrix} = \begin{pmatrix} 2 & -2 \\ 1 & 1 \end{pmatrix}$$

$$D = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} = \begin{pmatrix} 3 & 0 \\ 0 & -1 \end{pmatrix}$$

Påstod: $A \cdot P = P \cdot D$

$$P \cdot D = \begin{pmatrix} 2 & -2 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} 3 & 0 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} 6 & 2 \\ 3 & -1 \end{pmatrix}$$

$$A \cdot P = \begin{pmatrix} 1 & 4 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} 2 & -2 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} 6 & 2 \\ 3 & -1 \end{pmatrix}$$

$$D = \begin{pmatrix} -1 & 0 \\ 0 & 3 \end{pmatrix} \qquad P = \begin{pmatrix} -4 & -2 \\ 2 & -1 \end{pmatrix}$$

$$P \cdot D = \begin{pmatrix} -4 & -2 \\ 2 & -1 \end{pmatrix}\begin{pmatrix} -1 & 0 \\ 0 & 3 \end{pmatrix} = \begin{pmatrix} 4 & -6 \\ -2 & -3 \end{pmatrix}$$

$$A \cdot P = \begin{pmatrix} 1 & 4 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} -4 & -2 \\ 2 & -1 \end{pmatrix} = \begin{pmatrix} 4 & -6 \\ -2 & -3 \end{pmatrix}$$

Hvis P er <u>invertibel</u> kan vi da skrive $A = P \cdot D \cdot P^{-1}$. Da er A <u>diagonaliserbar</u>.

Eks: La $A = \begin{pmatrix} 1 & 4 \\ 1 & 1 \end{pmatrix}$

Regn ut A^3

$$A = P \cdot D \cdot P^{-1}$$

$$A^3 = A \cdot A \cdot A = P \cdot D \cdot \boxed{P^{-1} \cdot P} \cdot D \cdot \boxed{P^{-1} \cdot P} \cdot D \cdot P^{-1}$$

$$= P \cdot D \cdot I \cdot D \cdot I \cdot D \cdot P^{-1}$$

$$= P \cdot D^3 \cdot P^{-1}$$

$$= \begin{pmatrix} 2 & -2 \\ 1 & 1 \end{pmatrix} \cdot \begin{pmatrix} 3^3 & 0 \\ 0 & (-1)^3 \end{pmatrix} \begin{pmatrix} \frac{1}{4} & \frac{1}{2} \\ -\frac{1}{4} & \frac{1}{2} \end{pmatrix}$$

$$= \begin{pmatrix} 2 & -2 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 27 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} \frac{1}{4} & \frac{1}{2} \\ -\frac{1}{4} & \frac{1}{2} \end{pmatrix}$$

$$= \begin{pmatrix} 54 & 2 \\ 27 & -1 \end{pmatrix} \begin{pmatrix} \frac{1}{4} & \frac{1}{2} \\ -\frac{1}{4} & \frac{1}{2} \end{pmatrix} = \begin{pmatrix} \frac{54}{4} - \frac{2}{4} & \frac{54}{2} + 1 \\ \frac{27}{4} + \frac{1}{4} & \frac{27}{2} - \frac{1}{2} \end{pmatrix}$$

$$= \begin{pmatrix} 13 & 18 \\ 7 & 13 \end{pmatrix}$$

Invers for 2x2-matrisen

$$\begin{pmatrix} 2 & -2 \\ 1 & 1 \end{pmatrix}$$

① Finn determinanten
Her: 4

② Bytt plass på diagonal \
og fortegn på diagonal /

③ Del på determinanten.

$$\begin{pmatrix} 1 & 2 \\ -1 & 2 \end{pmatrix}$$

$$\begin{pmatrix} \frac{1}{4} & \frac{1}{2} \\ -\frac{1}{4} & \frac{1}{2} \end{pmatrix}$$

$$\left(\begin{array}{cc|cc} 2 & -2 & 1 & 0 \\ 1 & 1 & 0 & 1 \end{array}\right) \sim \left(\begin{array}{cc|cc} 1 & 0 & \frac{1}{4} & \frac{1}{2} \\ 0 & 1 & -\frac{1}{4} & \frac{1}{2} \end{array}\right)$$

Løs likningssystemet

$$\begin{pmatrix} 1 & 4 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -7 \\ -1 \end{pmatrix}$$

$$P^{-1}\vec{v} = \vec{u}$$

$$P^{-1}\vec{b} = \vec{c}$$

$$P \cdot D \cdot \boxed{P^{-1} \cdot \vec{v}} = \vec{b}$$

$$\boxed{D} \cdot \vec{u} = \vec{c}$$

$$\begin{pmatrix} \frac{1}{4} & \frac{1}{2} \\ -\frac{1}{4} & \frac{1}{2} \end{pmatrix}\begin{pmatrix} -7 \\ -1 \end{pmatrix} = \begin{pmatrix} -\frac{9}{4} \\ \frac{5}{4} \end{pmatrix}$$

$$\begin{pmatrix} 3 & 0 \\ 0 & -1 \end{pmatrix}\vec{u} = \begin{pmatrix} -\frac{9}{4} \\ \frac{5}{4} \end{pmatrix}$$

$$\begin{aligned} 3 \cdot x' &= -\frac{9}{4} \\ -y' &= \frac{5}{4} \end{aligned} \Rightarrow \begin{aligned} x' &= -\frac{3}{4} \\ y' &= -\frac{5}{4} \end{aligned}$$

$$P^{-1}\cdot\vec{v}=\vec{u}$$

$$P\cdot\vec{u}=\vec{v}$$

$$\begin{pmatrix} 2 & -2 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} -\frac{3}{4} \\ -\frac{5}{4} \end{pmatrix} = \begin{pmatrix} -\frac{3}{2}+\frac{5}{2} \\ -\frac{3}{4}-\frac{5}{4} \end{pmatrix} = \begin{pmatrix} 1 \\ -2 \end{pmatrix}$$

Vi definerte $x^{\circ}=\frac{1}{4}x+\frac{1}{2}y$

$$y^{\circ}=-\frac{1}{4}x+\frac{1}{2}y$$

Løste $D\cdot\vec{u}=\vec{c}$ i stedet for $A\cdot\vec{v}=\vec{b}$

$$\vec{u}=\begin{pmatrix} x^{\circ} \\ y^{\circ} \end{pmatrix} \qquad \vec{v}=\begin{pmatrix} x \\ y \end{pmatrix} \qquad \vec{c}=P^{-1}\cdot\vec{b}$$

Kan $\begin{pmatrix} 0 & -2 & 2 \\ -1 & 1 & 1 \\ -4 & -4 & 6 \end{pmatrix}$ diagonaliseres?

$$\begin{vmatrix} -\lambda & -2 & 2 \\ -1 & 1-\lambda & 1 \\ -4 & -4 & 6-\lambda \end{vmatrix} \underset{\sim}{R_1-\lambda R_2} \begin{vmatrix} 0 & -2-\lambda+\lambda^2 & 2-\lambda \\ -1 & 1-\lambda & 1 \\ -4 & -4 & 6-\lambda \end{vmatrix} \underset{\sim}{R_3-4R_2} \begin{vmatrix} 0 & \lambda^2-\lambda-2 & 2-\lambda \\ -1 & 1-\lambda & 1 \\ 0 & -8+4\lambda & 2-\lambda \end{vmatrix}$$

$$=1\cdot\begin{vmatrix} \lambda^2-\lambda-2 & 2-\lambda \\ 4\lambda-8 & 2-\lambda \end{vmatrix} = (\lambda^2-\lambda-2)(2-\lambda)-(2-\lambda)(4\lambda-8)$$

$$=(2-\lambda)(\lambda^2-\lambda-2-(4\lambda-8))$$

$$=(2-\lambda)(\lambda^2-5\lambda+6)$$

$(2-\lambda)(\lambda^2-5\lambda+6)=0$

$2-\lambda=0 \quad \Rightarrow \lambda=2$

$\lambda^2-5\lambda+6=0 \quad \Rightarrow \lambda=2$ og $\lambda=3$

Vi har „tre" egenverdier, $\lambda_1=2$, $\lambda_2=2$, $\lambda_3=3$

$$D=\begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix}$$

Egenvektor for $\lambda=3$

$$\begin{pmatrix} 0-\lambda & -2 & 2 \\ -1 & 1-\lambda & 1 \\ -4 & -4 & 6-\lambda \end{pmatrix} \vec{v} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\left(\begin{array}{ccc|c} -3 & -2 & 2 & 0 \\ -1 & -2 & 1 & 0 \\ -4 & -4 & 3 & 0 \end{array}\right) \underset{R_1\cdot(-1)}{\overset{R_1\to R_2}{\sim}} \left(\begin{array}{ccc|c} 1 & 2 & -1 & 0 \\ -3 & -2 & 2 & 0 \\ -4 & -4 & 3 & 0 \end{array}\right)$$

$$\underset{R_3+4R_1}{\overset{R_2+3R_1}{\sim}} \left(\begin{array}{ccc|c} 1 & 2 & -1 & 0 \\ 0 & 4 & -1 & 0 \\ 0 & 4 & -1 & 0 \end{array}\right) \underset{R_2\cdot\frac{1}{4}}{\overset{R_3-R_2}{\sim}} \left(\begin{array}{ccc|c} 1 & 2 & -1 & 0 \\ 0 & 1 & -\frac{1}{4} & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

$$x + 2y - z = 0 \qquad x = 2y$$
$$y - \frac{1}{4}z = 0 \qquad z = 4y$$
$$0 = 0$$

Velger $y=1$, får $\vec{v}_3 = \begin{pmatrix} 2 \\ 1 \\ 4 \end{pmatrix}$

Egenverdi $\lambda_1 = \lambda_2 = 2$

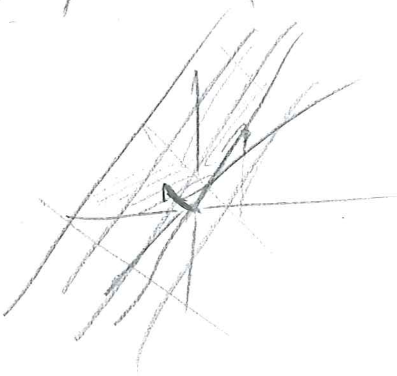
$$\begin{pmatrix} -2 & -2 & 2 & 0 \\ -1 & -1 & 1 & 0 \\ -4 & -4 & 4 & 0 \end{pmatrix} \underset{R_3 - 4R_2}{\overset{R_1 - 2R_2}{\sim}} \begin{pmatrix} 0 & 0 & 0 & 0 \\ -1 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\underset{R_2 \leftrightarrow R_1}{\overset{R_2 \cdot (-1)}{\sim}} \begin{pmatrix} 1 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$x + y - z = 0$$
$$0 = 0$$
$$0 = 0$$

$$x = -y + z$$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -y + z \\ y + 0z \\ 0y + z \end{pmatrix} = \begin{pmatrix} -y \\ y \\ 0 \end{pmatrix} + \begin{pmatrix} z \\ 0 \\ z \end{pmatrix} = y \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} + z \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$$



Alt 1: $y=1$, $z=0$ $\quad \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} = \vec{v}_1$

Alt 2: $y=0$, $z=1$ $\quad \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} = \vec{v}_2$

$$P = \begin{pmatrix} -1 & 1 & 2 \\ 1 & 0 & 1 \\ 0 & 1 & 4 \end{pmatrix}$$

$$D = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix}$$

Denne går alltid

$$\boxed{A \cdot P = P \cdot D}$$

$$A = P \cdot D \cdot P^{-1}$$

Denne krever at P er invertibel.

Vi trenger forskjellige retninger for egenvektorene.

<u>Eks:</u>

$$A = \begin{pmatrix} 3 & -2 & 1 \\ 1 & 0 & 1 \\ -1 & 1 & 2 \end{pmatrix}$$ har egenverdier $\lambda_1 = 1$, $\lambda_2 = 2$, $\lambda_3 = 2$

Om vi løser

$A \cdot \vec{v} = 2 \cdot \vec{v}$ får vi $x = y = z$

Egenvektor blir $t \cdot \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$ Kun én retning, så

klarer ikke å finne en invertibel P.

Så A er <u>ikke</u> diagonaliserbar.

Utregning siste (rakk ikke på forelesning)

$$A = \begin{pmatrix} 3 & -2 & 1 \\ 1 & 0 & 1 \\ -1 & 1 & 2 \end{pmatrix} \qquad |A-\lambda I| = \begin{vmatrix} 3-\lambda & -2 & 1 \\ 1 & -\lambda & 1 \\ -1 & 1 & 2-\lambda \end{vmatrix}$$

$$\begin{vmatrix} 3-\lambda & -2 & 1 \\ 1 & -\lambda & 1 \\ -1 & 1 & 2-\lambda \end{vmatrix} \overset{R_3+R_2}{\sim} \begin{vmatrix} 3-\lambda & -2 & 1 \\ 1 & -\lambda & 1 \\ 0 & 1-\lambda & 3-\lambda \end{vmatrix} \overset{R_1-(3-\lambda)R_2}{\sim} \begin{vmatrix} \oplus 0 & -\lambda^2+3\lambda-2 & \lambda-2 \\ \ominus 1 & -\lambda & 1 \\ \oplus 0 & 1-\lambda & 3-\lambda \end{vmatrix}$$

$$= -1 \cdot \begin{vmatrix} -\lambda^2+3\lambda-2 & \lambda-2 \\ 1-\lambda & 3-\lambda \end{vmatrix} = -1\cdot\left(\left(-\lambda^2+3\lambda-2\right)(3-\lambda) - (\lambda-2)(1-\lambda)\right)$$

$$= -1\cdot\left(\lambda^3 - 3\lambda^2 - 3\lambda^2 + 9\lambda + 2\lambda - 6 - \left(\lambda - \lambda^2 - 2 + 2\lambda\right)\right)$$

$$= -\lambda^3 + 5\lambda^2 - 8\lambda + 4 = 0$$ Løser denne (for eksempel kalkulator) og får $\lambda_1 = 1,\ \lambda_2 = 2,\ \lambda_3 = 2.$

For $\lambda = 2$ får vi da egenvektor ved:

$$\begin{pmatrix} 3-2 & -2 & 1 \\ 1 & 0-2 & 1 \\ -1 & 1 & 2-2 \end{pmatrix} = \begin{pmatrix} 1 & -2 & 1 \\ 1 & -2 & 1 \\ -1 & 1 & 0 \end{pmatrix} \overset{R_2-R_1}{\sim} \begin{pmatrix} 1 & -2 & 1 \\ 0 & 0 & 0 \\ -1 & 1 & 0 \end{pmatrix}$$

$$\overset{R_3+R_1}{\sim} \begin{pmatrix} 1 & -2 & 1 \\ 0 & 0 & 0 \\ 0 & -1 & 1 \end{pmatrix} \overset{R_1-2R_3}{\sim} \begin{pmatrix} 1 & 0 & -1 \\ 0 & 0 & 0 \\ 0 & -1 & 1 \end{pmatrix}$$

som gir $x = z$, $y = z$. Én fri variabel, ikke nok til å lage to uavhengige egenvektorer.