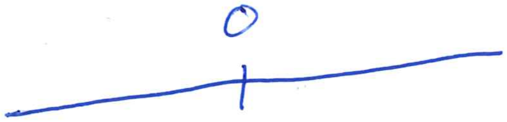
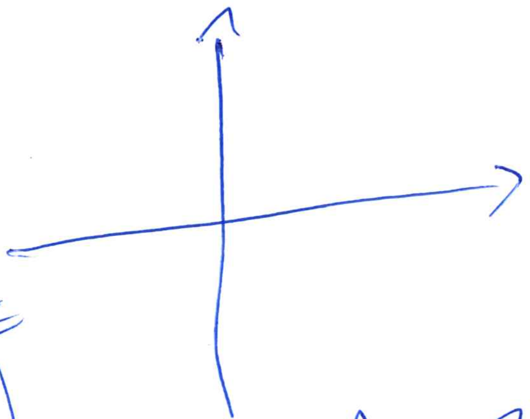
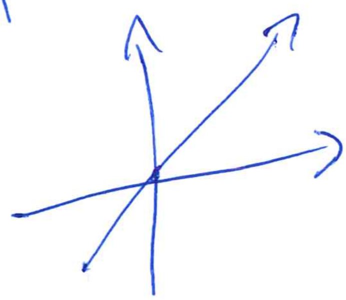
Lineære transformasjoner

Notasjon

$x \in$  $- \mathbb{R}$ - de reelle tallene
- tallinja

$\begin{pmatrix} x \\ y \end{pmatrix} \in$  $\mathbb{R}^2$ - planet

$\begin{pmatrix} x \\ y \\ z \end{pmatrix} \in$  $\mathbb{R}^3$ - rommet

$\begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix} \in$??? $\mathbb{R}^4$ - 4D-rom

Funksjoner

En funksjon tar inn ett tall, og gir ut ett tall.

$$f(x) = 3x^2 - 2 \qquad 2 \mapsto 10$$

$$f(2) = 3 \cdot 2^2 - 2 = 10$$

Hvorfor stoppe med ett tall?

$$f(x, y, z) = \begin{pmatrix} 3x^2 + 2y - z \\ 2y + z^3 \end{pmatrix}$$

$$f(3, 5, -1) = \begin{pmatrix} 38 \\ 9 \end{pmatrix} \qquad \begin{pmatrix} 3 \\ 5 \\ -1 \end{pmatrix} \mapsto \begin{pmatrix} 38 \\ 9 \end{pmatrix}$$

Skriver $\quad f : \mathbb{R}^3 \to \mathbb{R}^2$

tar inn tre tall — Gir ut to tall

"Vanlige" funksjoner:

$$f : \mathbb{R} \to \mathbb{R}$$

En lineær transformasjon:

Er en funksjon $f : \mathbb{R}^n \to \mathbb{R}^m$ som:

1. $f(\vec{0}) = \vec{0}$ $\qquad \vec{0} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$

2. Linjer blir til linjer
Plan blir til plan
etc.

En alternativ, lettere å sjekke, definisjon:

1. $f(c \cdot \vec{v}) = c \cdot f(\vec{v})$

2. $f(\vec{u} + \vec{v}) = f(\vec{u}) + f(\vec{v})$

Eks: Er funksjonen fra i stad en lineærtransformasjon?

$$f(x,y,z) = \begin{pmatrix} 3x^2 + 2y - z \\ 2y + z^3 \end{pmatrix}$$

Test $\vec{u} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$ $\qquad \vec{v} = \begin{pmatrix} 2 \\ 2 \\ 0 \end{pmatrix}$

$$f(\vec{u}) = \begin{pmatrix} 4 \\ 3 \end{pmatrix} \qquad f(\vec{v}) = \begin{pmatrix} 16 \\ 4 \end{pmatrix}$$

$$f(\vec{u} + \vec{v}) = f\left(\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} + \begin{pmatrix} 2 \\ 2 \\ 0 \end{pmatrix}\right) = f\begin{pmatrix} 3 \\ 3 \\ 1 \end{pmatrix} = \begin{pmatrix} 32 \\ 7 \end{pmatrix}$$

$\#$

$$f(\vec{u}) + f(\vec{v}) = \begin{pmatrix} 4 \\ 3 \end{pmatrix} + \begin{pmatrix} 16 \\ 4 \end{pmatrix} = \begin{pmatrix} 20 \\ 7 \end{pmatrix}$$

$$f(x,y) = \begin{pmatrix} 2x - y \\ x + 3y \\ -x - 2y \end{pmatrix}$$

Må vise:

$$f(c \cdot \vec{u}) = c \cdot f(\vec{u}) \qquad \vec{u} = \begin{pmatrix} x \\ y \\ \cancel{z} \end{pmatrix}$$

$$f\left(c \cdot \begin{pmatrix} x \\ y \\ \cancel{z} \end{pmatrix}\right) = f\begin{pmatrix} cx \\ cy \\ \cancel{cz} \end{pmatrix} = \begin{pmatrix} 2 \cdot c \cdot x - c \cdot y \\ c \cdot x + 3 \cdot c \cdot y \\ -c \cdot x - 2 \cdot c \cdot y \end{pmatrix}$$

$$= \begin{pmatrix} c \cdot (2x - y) \\ c \cdot (x + 3y) \\ c \cdot (-x - 2y) \end{pmatrix} = c \cdot \begin{pmatrix} 2x - y \\ x + 3y \\ -x - 2y \end{pmatrix} = c \cdot f(\vec{u})$$

$$f(\vec{u} + \vec{v}) = f(\vec{u}) + f(\vec{v}) \qquad \vec{u} = \begin{pmatrix} x \\ y \end{pmatrix} \quad \vec{v} = \begin{pmatrix} z \\ w \end{pmatrix}$$

$$f\left(\begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} z \\ w \end{pmatrix}\right) = f\begin{pmatrix} x + z \\ y + w \end{pmatrix} = \begin{pmatrix} 2(x+z) - (y+w) \\ (x+z) + 3(y+w) \\ -(x+z) - 2(y+w) \end{pmatrix}$$

$$= \begin{pmatrix} 2x - y + 2z - w \\ x + 3y + z + 5w \\ -x - 2y - z - 2w \end{pmatrix} = \begin{pmatrix} 2x - y \\ x + 3y \\ -x - 2y \end{pmatrix} + \begin{pmatrix} 2z - w \\ z + 3w \\ -z - 2w \end{pmatrix}$$

$$= f\begin{pmatrix} x \\ y \end{pmatrix} + f\begin{pmatrix} z \\ w \end{pmatrix}$$

Matrisen som funksjoner

$$\begin{array}{cc} & 3 \\ 2 & \begin{pmatrix} 2 & -1 & 3 \\ 5 & 1 & 0 \end{pmatrix} \end{array} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 2x - y + 3z \\ 5x + y \end{pmatrix}$$

Se på dette som en <u>funksjon</u> $f: \mathbb{R}^3 \to \mathbb{R}^2$

Teorem:

$f(\vec{u}) = A \cdot \vec{u}$ så er f en lineær transformasjon.

$$f(c \cdot \vec{u}) = A \cdot (c \cdot \vec{u}) = c \cdot A \vec{u} = c \cdot f(\vec{u})$$

$$f(\vec{u} + \vec{v}) = A(\vec{u} + \vec{v}) = A \cdot \vec{u} + A \cdot \vec{v} = f(\vec{u}) + f(\vec{v})$$

Hvis f er en lineær transformasjon så <u>må</u> det finnes en matrise A slik at

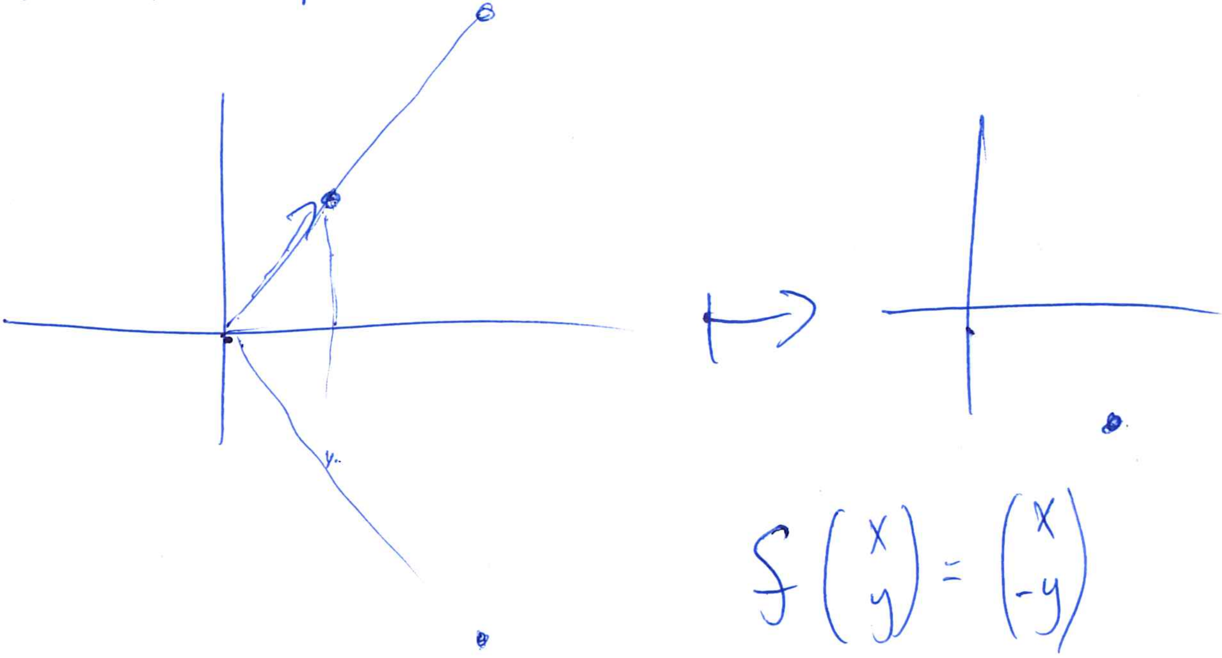
$$f(\vec{u}) = A \cdot \vec{u}$$

Eks:

$$f\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 2x - y \\ x + 3y \\ -x - 2y \end{pmatrix} = \begin{pmatrix} 2 & -1 \\ 1 & 3 \\ -1 & -2 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$$

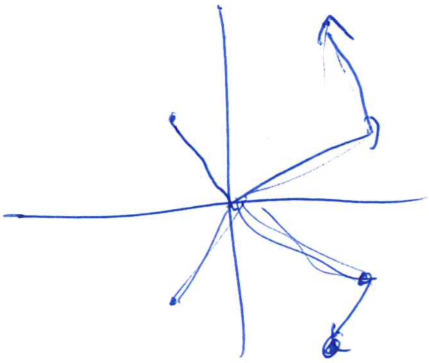
Har funksjon $f(x,y)$, $f: \mathbb{R}^2 \to \mathbb{R}^2$

gitt ved å speile langs x-aksen.



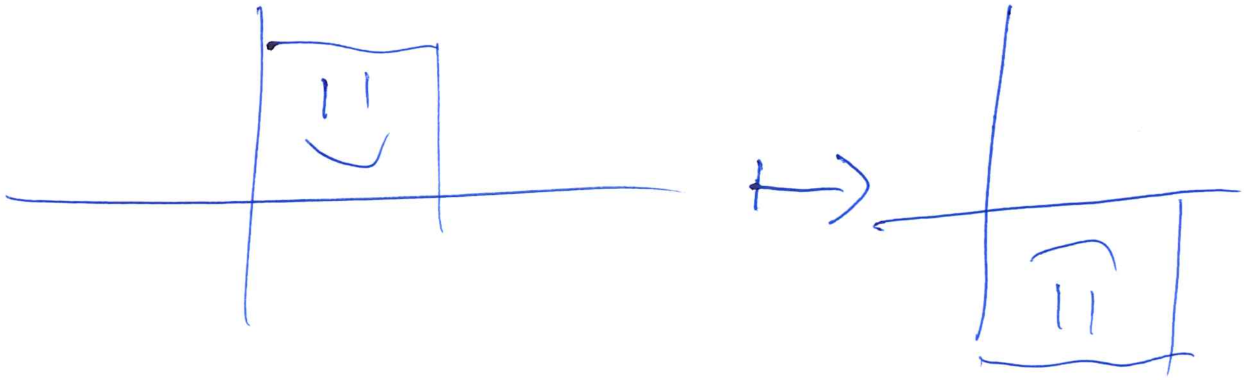
$$f\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x \\ -y \end{pmatrix}$$

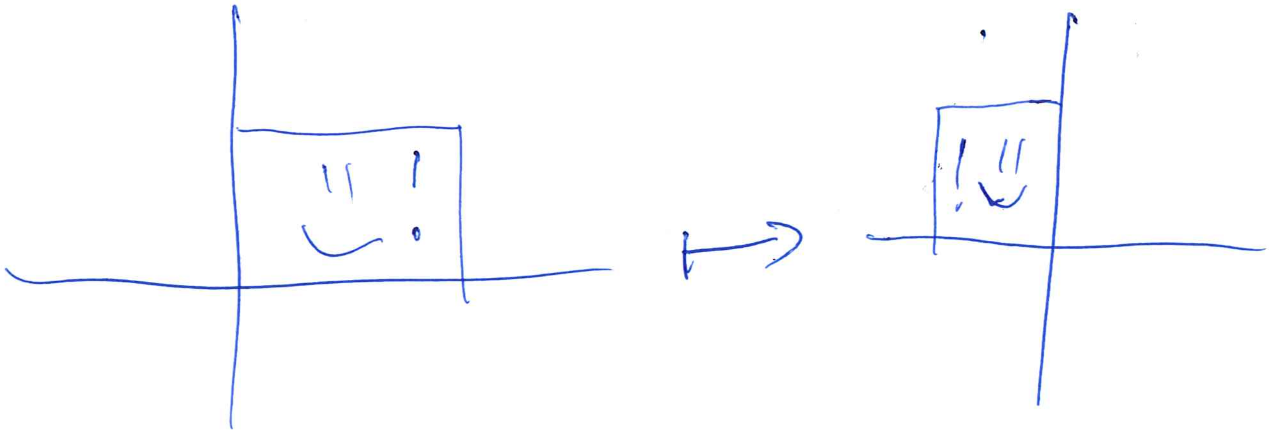
$$= \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$$

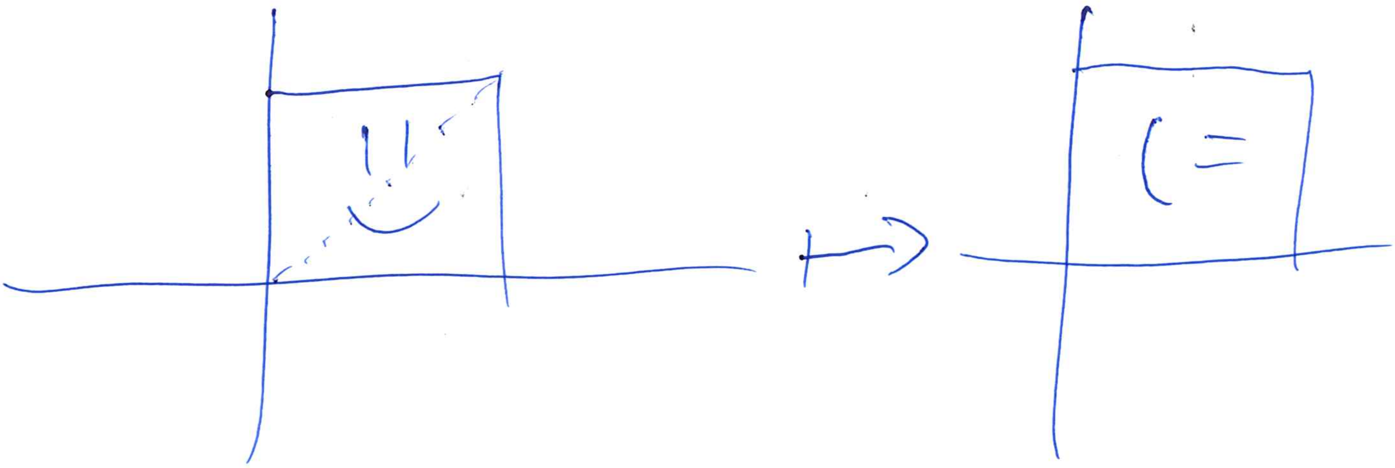


Lineærtransformasjoner fra $\mathbb{R}^2$ til $\mathbb{R}^2$

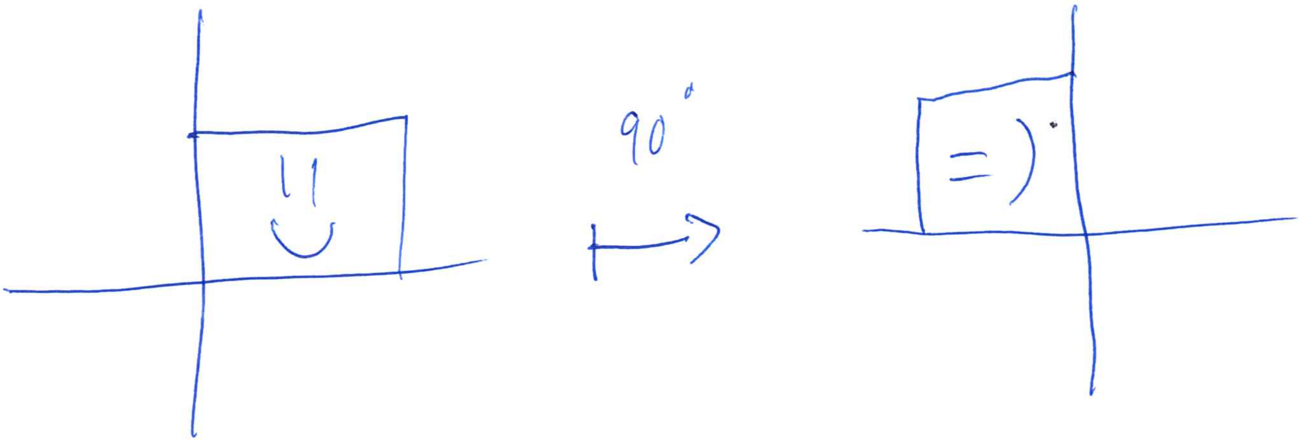
Speiling langs en linje

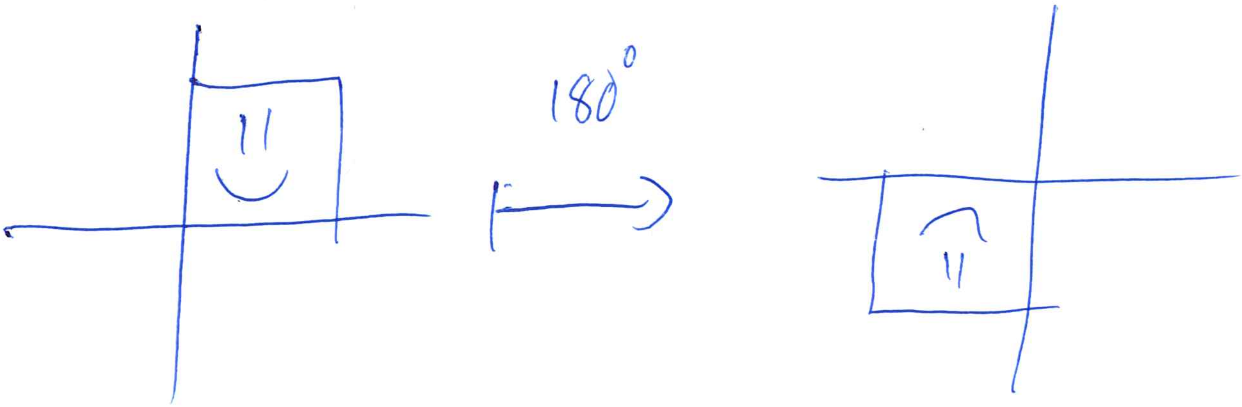


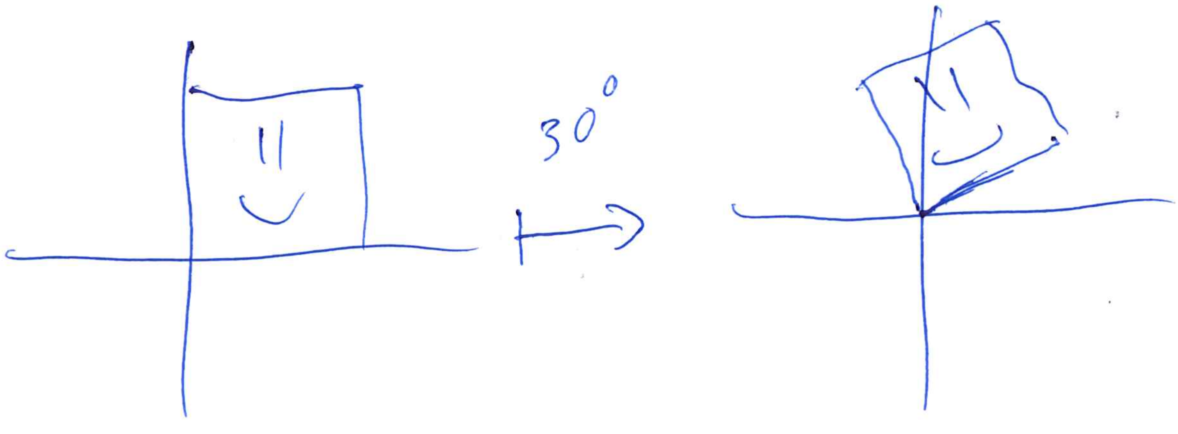




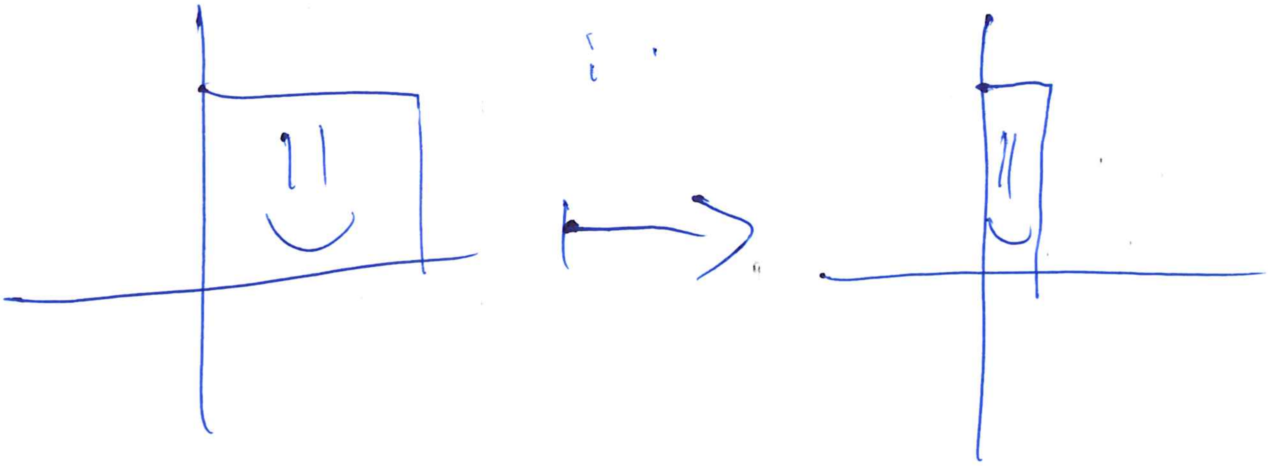
Rotasjoner om origo

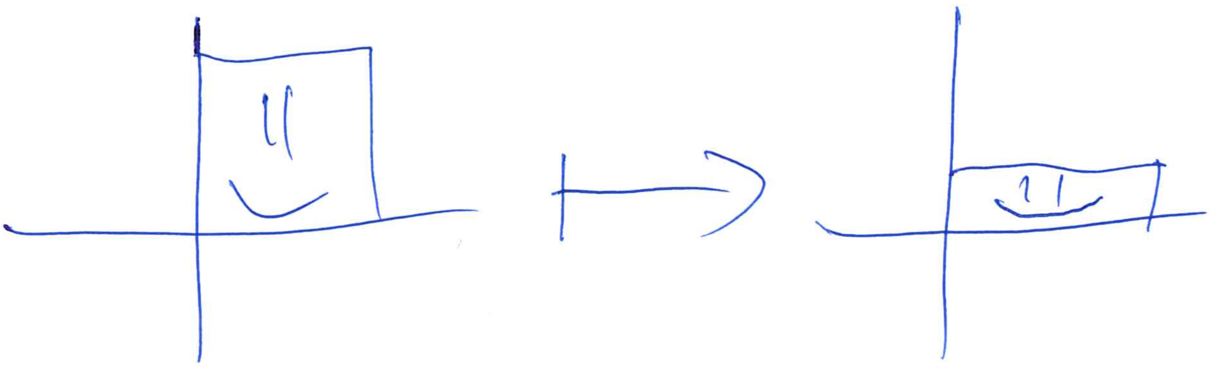


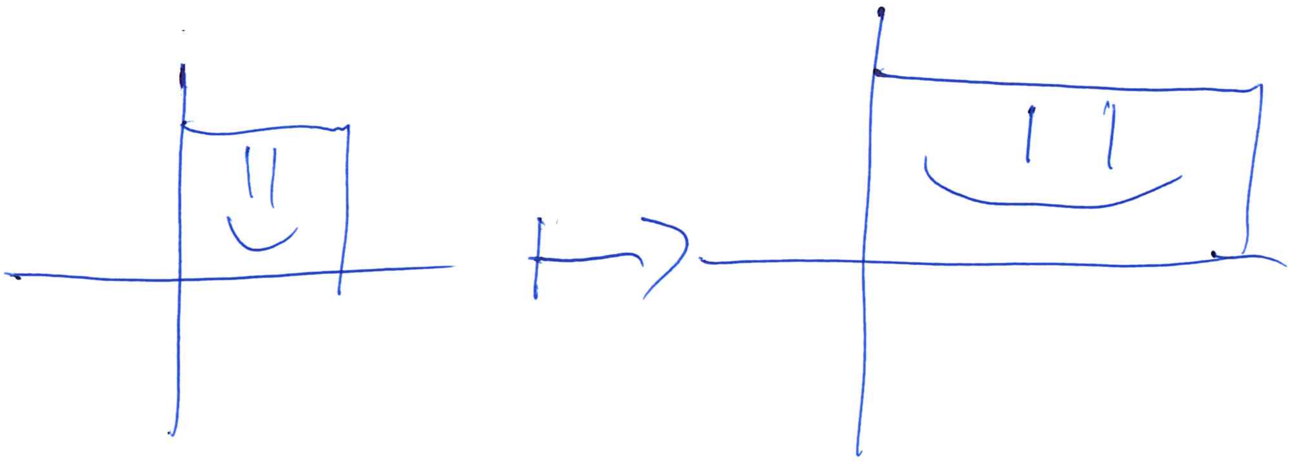




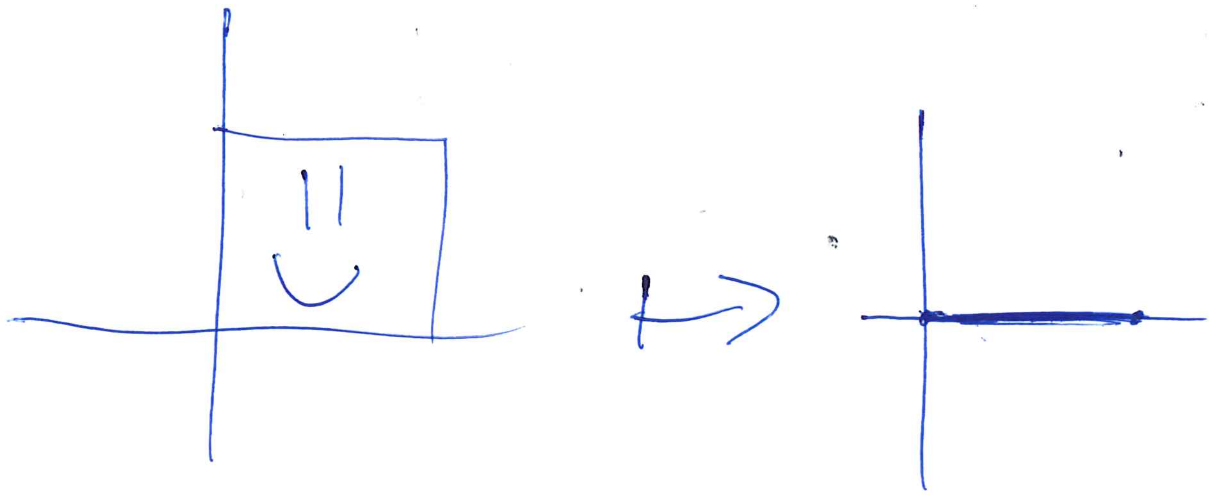
Skalering i en retning



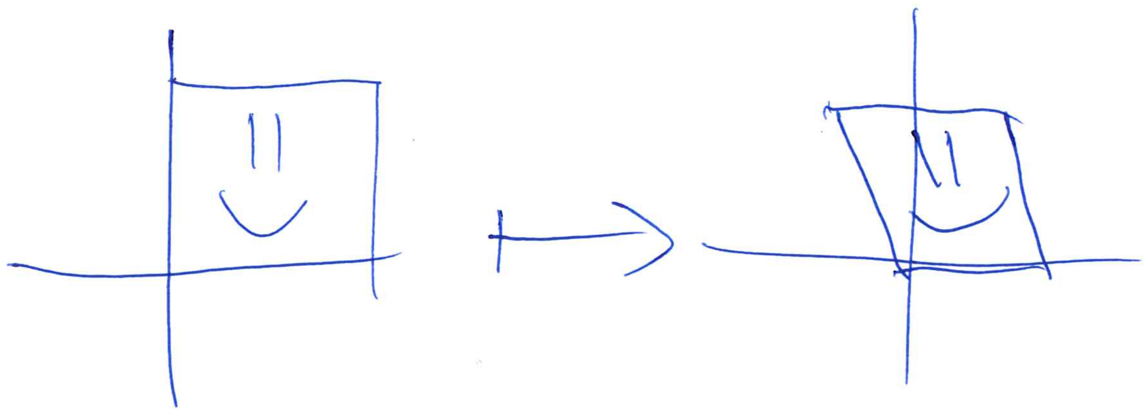




Projeksjon på linje



Skjærtransformasjon



Matrisene til rotasjoner:

Hvis $f\begin{pmatrix} x \\ y \end{pmatrix}$ roterer θ radianer om origo, er matrisen gitt ved

$$\begin{pmatrix} \cos\theta, & -\sin\theta \\ \sin\theta, & \cos\theta \end{pmatrix}$$

Eks: Rotasjon med 30°:

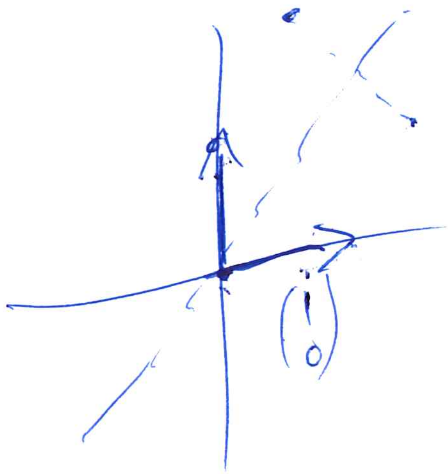
$$\begin{pmatrix} \cos 30^\circ & -\sin 30^\circ \\ \sin 30^\circ & \cos 30^\circ \end{pmatrix} = \begin{pmatrix} \frac{\sqrt{3}}{2} & -\frac{1}{2} \\ \frac{1}{2} & \frac{\sqrt{3}}{2} \end{pmatrix}$$

Generell regel for lineærtransformasjoner:

Hvis $f\begin{pmatrix} x \\ y \end{pmatrix}$ er en lineærtransformasjon så er matrisen til f gitt ved

$$\begin{pmatrix} f\begin{pmatrix} 1 \\ 0 \end{pmatrix} & f\begin{pmatrix} 0 \\ 1 \end{pmatrix} \end{pmatrix}$$

Eks: ~~f(x)~~ $f\begin{pmatrix}x\\y\end{pmatrix}$ er speiling langs linja $x=y$



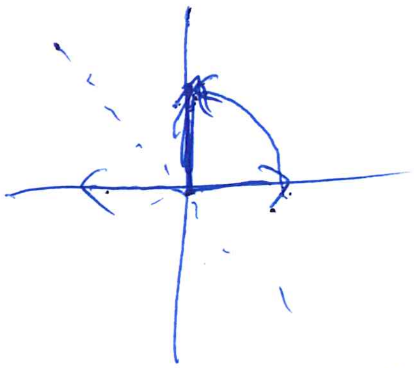
$$\begin{pmatrix}1\\0\end{pmatrix} \mapsto \begin{pmatrix}0\\1\end{pmatrix}$$

$$\begin{pmatrix}0\\1\end{pmatrix} \mapsto \begin{pmatrix}1\\0\end{pmatrix}$$

$$A = \left(f\begin{pmatrix}1\\0\end{pmatrix} \quad f\begin{pmatrix}0\\1\end{pmatrix} \right) = \begin{pmatrix}0 & 1\\1 & 0\end{pmatrix}$$

Eks: $f\begin{pmatrix}x\\y\end{pmatrix}$ roterer 90° og så speiler langs

$x=-y$

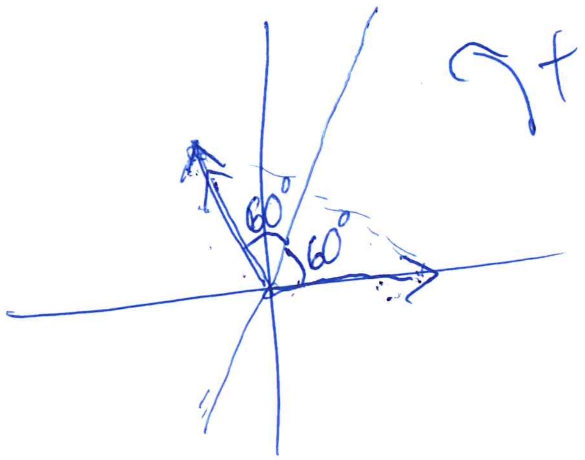


$$\begin{pmatrix}1\\0\end{pmatrix} \mapsto \begin{pmatrix}-1\\0\end{pmatrix}$$

$$\begin{pmatrix}0\\1\end{pmatrix} \mapsto \begin{pmatrix}0\\1\end{pmatrix}$$

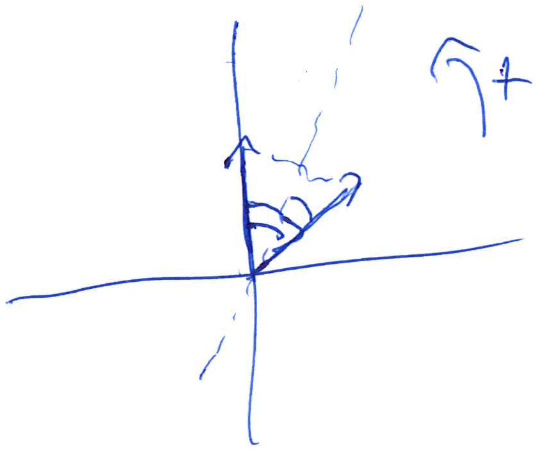
$$A = \left(f\begin{pmatrix}1\\0\end{pmatrix} \quad f\begin{pmatrix}0\\1\end{pmatrix} \right) = \begin{pmatrix}-1 & 0\\0 & 1\end{pmatrix}$$

$f\begin{pmatrix}x\\y\end{pmatrix}$ er speiling langs linja $y = \cancel{2x}$ $y = \sqrt{3}x$



$$\begin{pmatrix}\cos 120^\circ & -\sin 120^\circ \\ \sin 120^\circ & \cos 120^\circ\end{pmatrix}\begin{pmatrix}1\\0\end{pmatrix}$$

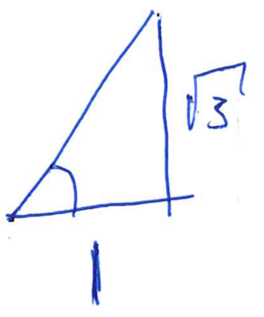
$$= \begin{pmatrix}\cos 120^\circ \\ \sin 120^\circ\end{pmatrix} = \begin{pmatrix}-\frac{1}{2} \\ \frac{\sqrt{3}}{2}\end{pmatrix}$$



$$\begin{pmatrix}\cos -60^\circ & -\sin -60^\circ \\ \sin -60^\circ & \cos -60^\circ\end{pmatrix}\cdot\begin{pmatrix}0\\1\end{pmatrix}$$

$$= \begin{pmatrix}-\sin(-60^\circ) \\ \cos(-60^\circ)\end{pmatrix} = \begin{pmatrix}\frac{\sqrt{3}}{2} \\ \frac{1}{2}\end{pmatrix}$$

$$A = \left(f\begin{pmatrix}1\\0\end{pmatrix} \quad f\begin{pmatrix}0\\1\end{pmatrix}\right) = \begin{pmatrix}-\frac{1}{2} & \frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & \frac{1}{2}\end{pmatrix}$$



$$\tan v = \frac{\sqrt{3}}{1} = \sqrt{3}$$

$$v = 60^\circ$$

I flere dim:

Matrisen til $f: \mathbb{R}^n \to \mathbb{R}^m$

er gitt ved

$$A = \left(f\begin{pmatrix}1\\0\\\vdots\\0\end{pmatrix}\ f\begin{pmatrix}0\\1\\\vdots\\0\end{pmatrix}\ f\begin{pmatrix}0\\0\\1\\\vdots\\0\end{pmatrix} \cdots f\begin{pmatrix}0\\0\\\vdots\\1\end{pmatrix}\right)$$

Kombinasjon av lineær transformasjoner

Hvis
$$\begin{tikzcd} \mathbb{R}^n \arrow[r, "f"] \arrow[rr, bend right] & \mathbb{R}^m \arrow[r, "g"] & \mathbb{R}^k \end{tikzcd}$$

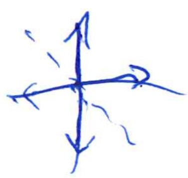
$f(\vec{u}) = A \cdot \vec{u}$ $\qquad$ $g(f(\vec{u})) = B \cdot A \cdot \vec{u}$

$g(\vec{v}) = B \cdot \vec{v}$

Eks: f er rotasjon med 90° så speiling langs $y=-x$

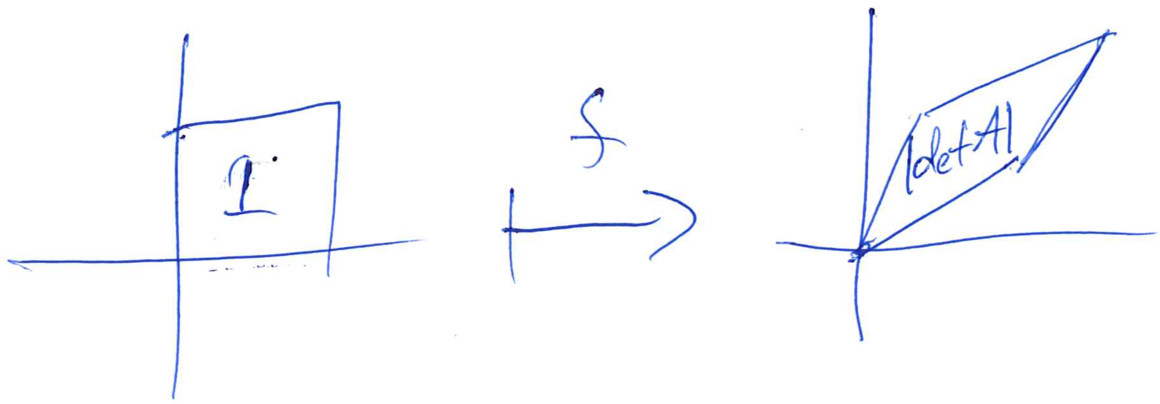
$$\begin{pmatrix}0 & -1\\1 & 0\end{pmatrix} \qquad \begin{pmatrix}0 & -1\\-1 & 0\end{pmatrix}$$

$$\begin{pmatrix}0 & -1\\-1 & 0\end{pmatrix} \cdot \begin{pmatrix}0 & -1\\1 & 0\end{pmatrix} = \begin{pmatrix}-1 & 0\\0 & 1\end{pmatrix}$$



Determinanter

Determinanten til en matrise forteller hvor mye arealet endrer seg under transformasjon



Ditto volum i 3D.

Derfor:
$$\begin{vmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{vmatrix} = 1$$

$$\cos^2\theta + \sin^2\theta = 1$$